Wave propagation in structures with Sturmian quasiperiodic patterns

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Abstract
Quasiperiodic systems are characterized by deterministic patterns that exhibit correlation in the long-range order and present unique properties that have been the subject of extensive research in various areas of physics as electronics, electromagnetics or elasticity. One of the most visible features of quasiperiodicity is the self-similarity of the spectrum of permitted modes and frequencies. Sturmian quasiperiodic lattices are under consideration in this work. A Sturmian word (or sequence) is a particular case of an infinite word formed from a two-letter alphabet and, among other applications, allows the construction of quasiperiodic patterns. Such sequences, also called mechanical words, can be generated from real numbers. Here we propose to investigate systems structured according to quasiperiodic patterns governed by so-called Sturmian sequences. This research is carried out in the context of structural dynamics and allows the application to mechanical engineering of concepts that until now have been applied in other fields.

1 Introduction

The manipulation of waves is the main topic in many engineering areas. For doing that, the inverse design of structured media with tailored properties is an important tool. In the last decades the research for using periodic systems has been a very intensive area of work. However, periodic media have some limitations and other types of symmetries are worthwhile to be explored. Quasi-periodic systems are promising candidates and they have been less well studied. They are characterized by deterministic patterns that exhibit correlation in the long-range order and present unique properties that have been the subject of extensive research in various areas of physics: electronics [1], electromagnetis [2], elasticity [3, 4]. The dispersion relation of these systems exhibit band gaps, as the periodic ones. The most celebrated property of these media is the self-similarity of the spectrum of permitted modes and frequencies [5, 6], as for example, in systems constructed with the Fibonacci formalism [7, 8], which often take the form of the well-known Hofstadter butterfly [9]. In mechanics, despite a few studies, the dispersion properties of quasiperiodic elastic media have not yet been satisfactorily understood, specially the design of tailored structures with specific properties concerning wave propagation. In references [10, 11, 12, 13] the elastodynamical properties of finite or infinite periodic 1D rods and beams are studied. Recent developments of the research in this field comprise the analysis of how localized modes arise in continuous elastic media with quasiperiodic stiffness modulation [4] of the analysis of the effects of combined modulation of structural parameters with different arbitrarily related spatial periods on wave propagation properties of a general 1D waveguide [14]. In ref. [15] the investigation of the frequency spectrum of 1D continuous quasiperiodic elastic media shows the fractal nature of the occurrence of bandgaps. In the literature we can find different examples on the study of the spectrum of allowed modes as a function of a certain parameter that generates the quasiperiodic pattern. The most relevant case is that of the aforementioned Hoesteadtler butterfly [9] in condensed matter, which has been reproduced and studied in other works [8, 16]. Following the generation pattern known as the projection method [17] other authors have obtained similar figures in other physical systems like discrete mass-spring lattices [18], quasiperiodic beams [4], dielectric quasicrystals [19] or acoustic metamaterials [20].

Here we propose to investigate systems which are structured according to quasiperiodic patterns governed by the so-called Sturmian sequences, something that will be carried out in the context of structural dynamics. First we will introduce Sturmian sequences, and how to implement structured systems based on them. We will also introduce the concept of Sturmian bulk spectrum and study the observed self-similarity properties. Finally, two numerical examples are given in order to validate the theoretical results. In this sense, the application of Sturmian sequences to mechanical engineering is revealed as a tool for the design of mechanical structures with tailored properties for wave propagation.

2 Sturmian Words

In this section we will describe precisely how to build Sturmian sequences. These will be generated from a real number, denoted by $\alpha \in \mathbb{R}$ that plays the role of *generation parameter* and, without loss of generality, it lies in the range $0 \leq \alpha \leq 1$.

Considere the sequence $[0; a_1, \ldots, a_n]$, with $a_k > 0$, for $k \geq 1$ positive integer numbers as the continuous fraction of α, namely

$$\alpha = [0; a_1, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} \,, \tag{1}$$

Consider in addition a binary alphabet formed by two symbols, say $\{p, q\}$. Then, we define a Sturmian word in a recursively way as the sequence of symbols

$$\begin{aligned} \mathcal{B}_k &= \mathcal{B}_{k-1}^{a_k}\, \mathcal{B}_{k-2} \,, \quad 1 \leq k \leq n \,, \\ \mathcal{B}_{-1} &= q \,, \quad \mathcal{B}_0 = p \,, \end{aligned} \tag{2}$$

where both the exponent and the product must be understood as concatenations, i.e., $p^3(q^2p) = pppqqp$. When α is a rational number, there exist a maximum numbrer fo iterations n being the Sturmian word $\mathcal{B}_n$ the last one and the infinite word becomes in a periodic concatenation of $\mathcal{B}_n$. On the contrary, if α is irrational, then it is known that the sequence a_n becomes infinite and the associated Sturmian word has a purely quasiperiodic pattern given by $\lim_{n\to\infty} \mathcal{B}_n$. This form of constructing a Sturmian word is not unique [21, 22] and also have different geometrical interpretations and recursive models [23, 24].

Each one of the words emerging from the recursive sequence (2) will be named *Sturmian blocks*. The last block of a sequence $\{\mathcal{B}_k\}_{k=1}^n$, is said to be the Sturmian block associated to α, and for them we will use the notation $\mathcal{B}(\alpha) = \mathcal{B}_n$. For numerical purposes, irrational numbers must be approximated by rationals approximants.

Let us denote by $\mathcal{N}_k$ to the total number of symbols of the kth block, for $k \geq 0$. Due to the recursive relation of Eq. (2) in which at each step new a_k blocks ot type $\mathcal{B}_{k-1}$ are added to the existing block $\mathcal{B}_{k-2}$, it follows immediately that

$$\mathcal{N}_k = a_k \mathcal{N}_{k-1} + \mathcal{N}_{k-2} \,, \quad 1 \leq k \leq n \,, \qquad \mathcal{N}_{-1} = 1 \,, \quad \mathcal{N}_0 = 1 \,. \tag{3}$$

We are going to pay attention to the extreme values of α, 0 and 1. The value $\alpha = 0$ does not have strictly a continued fraction as shown in Eq. (1) but can be considered the limit of $0 = \lim_{r\to\infty}[0; r]$ and therefore its associated sequence will also be the limit

$$\mathcal{B}(\alpha = 0) = \lim_{r\to\infty} \mathcal{B}(1/r) = \lim_{r\to\infty} p^r\, q = ppppp \ldots \tag{4}$$

On the other side, the value $\alpha = 1$ has as (degenerated) continuous fraction $1 = 1/1$ and therefore its associated Sturmian sequence is

$$\mathcal{B}(\alpha = 1) = pq\,pq\,pq\,pq\ \ldots \tag{5}$$

Both limit values $\alpha = 0$ and 1 correpond, in terms of a physical systems, with two well known structures: a homogeneous medium and the periodic bi-layered structure, respectively.

3 Systems with quasiperiodic distribution of physical parameters based on Sturmian words

In the previous section we have described the formation of words based on Sturmian sequences. In this section we are going to explain how we can form different types of mechanical systems based on this idea, the quasi-periodic variation of certain physical parameters. Consider a 1D dynamical system that we are going to build it via the concatenation of different elements as for instance masses, springs, rods, elastic supports, beams.... All of these elements have mechanical, inertial and geometrical properties in the context of elastic waves.

Let us start with our first example consisting in a discrete lumped mass system. This system is characterized by the parameters mass and spring coefficient. Given a number $\alpha \in [0, 1] = [0; a_1, \ldots, a_n]$, then the Sturmian block associated to α, $\mathcal{B}(\alpha) = \mathcal{B}_n$, has exactly $N = \mathcal{N}_n$ symbols according to the pattern given by Eqs. (2) and (3). We can build our system by the periodic repetition of the block $\mathcal{B}(\alpha)$, which in turn is formed by N elements. One of the parameters mentinoned above (mass or spring constant,...), that we call generically Θ, can vary but taking only two values among the binary set $\{\theta_p, \theta_q\}$. The rest of parameters remain constant from element to element along the block. Thus, if $\Theta(j)$ denotes the value of the parameter of the jth element, with $1 \le j \le N$, then we have

$$\Theta(j) = \begin{cases} \theta_p & \text{if the } j\text{th term of } \mathcal{B}(\alpha) \text{ is } p \\ \theta_q & \text{if the } j\text{th term of } \mathcal{B}(\alpha) \text{ is } q \end{cases}, \quad 1 \le j \le N\ . \tag{6}$$

In the Fig. 1 the building process of the system is illustrated with three examples, a discrete spring-mass system, a continuous rod (axial waves) and a continuous beam (flexural waves).

The three systems share the number $\alpha = 2/7 = [0; 2, 3]$, resulting in the block $\mathcal{B}(\alpha) = pppqpppqp$ and in consequence, all of them are instances of a quasiperiodic Sturmian pattern. In the spring-mass system, the mass remains fixed but the rigidity of the spring assumes the roll of the parameter, i.e. $\Theta \equiv k$, and k_p or k_q are depending on the Sturmian sequence within $\mathcal{B}(\alpha)$. The second system (shown in the middle of Fig. 1) represents a straight rod with density ρ, cross sectional area A and Young modulus E. As known, the axial compressional waves propagate at a velocity $\sqrt{EA/\rho A}$. The infinite medium is structured into elements of length l. In the particular case of this example, the axial stiffness EA is constant meanwhile the mass per unit of length ρA varies among two values $\{\rho A_p, \rho A_q\}$ as indicated in the the Sturmian block. Finally the third example represents a beam on simple supports. Possible parameters which can be assigned to Θ are, for instance, $\{\rho A, EI, GA_s\}$, where EI and GA_s are the sectional bending and shear stiffness, respectively. Note that even the span length between supports could also be changed from element to element obeying the Sturmian block pattern. In Fig. 1 (bottom) the bending stiffness is assumed to take one of the two values EI_p and EI_q as prescribed in $\mathcal{B}(\alpha)$. In this case, the three examples have $N = 9$ elements which are repeated periodically.

If α is an irrational number, theoretically the system is not periodic because $\mathcal{B}(\alpha)$ has infinite number of symbols. In a real case, α has to be approximated by the nth convergent [24]. As n increases, the effects of quasiperiodicity become more relevant and visible in the system. One of the consequences is the selfsimilarity of the spectrum as more terms of the sequence $\{a_n\}$ are added, something that can be visualized for the Fibonacci case in ref. [25]

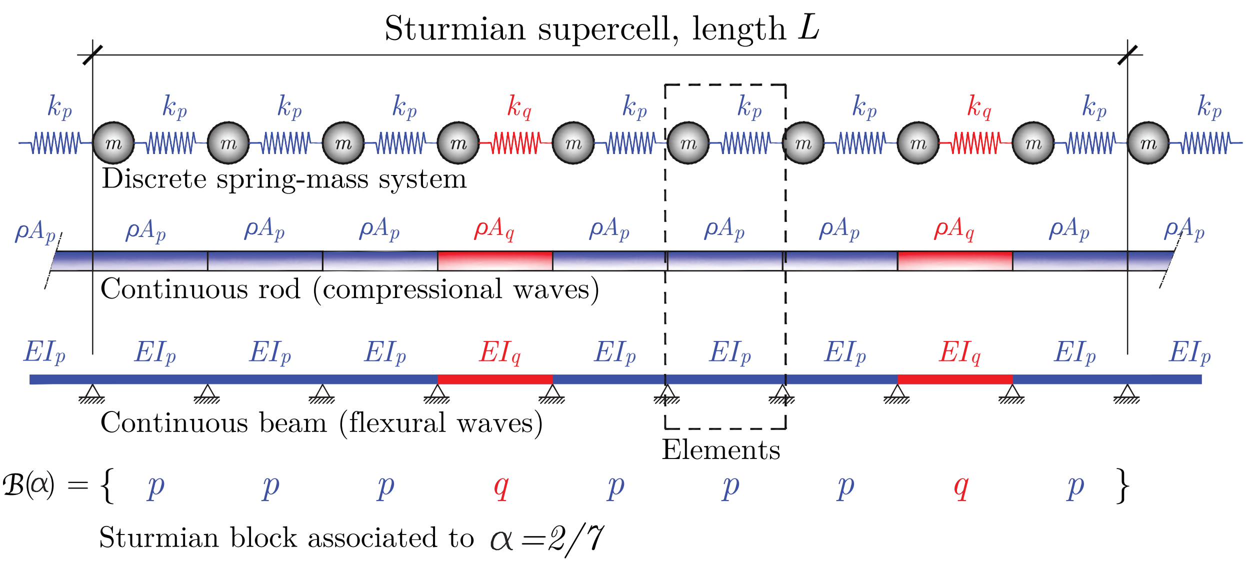

Figure 1: Three simple examples of how we can build a dynamical systems based on Sturmian blocks correspondint to number $\alpha = 2/7 = [0; 3, 2]$. Above: a discrete spring-mass system, $\Theta \equiv k$ (spring coefficients). Middle: a continuous rod (axial waves), $\Theta \equiv \rho A$ (mass per unit of length). Bottom: a continuous beam (flexural waves), $\Theta \equiv EI$ (sectional bending stiffness)

In practice we can only built systems associated to rational numbers (finite aproximants in case α to be irrational), the study of such kind of systems is done via the use of the so-called *supercell* with $N = \mathcal{N}_n$ elements. Since the system is formed by periodic repetition of the Sturmian block $\mathcal{B}(\alpha)$, then $\Theta(j + N) = \Theta(j)$ for $j > N$. This *supercell* will be repeated periodically in the same way as the associated Sturmian word, forming a mechanical waveguide. This will allow us to study the dispersion properties of such systems.

4 Spectrum of Sturmian structured media

It is well-known that one of the most relevant properties of wave propagation in periodic media is the emergence of bandgaps in the frequency spectrum. A proper design of the unit cell can result in optimized location of bandgaps or passbands. On other side, quasiperiodic media, like for example Fibonacci sequence–based systems, exhibit self-similarity of the spectrum [5, 6, 7]. Here we want to study the spectral properties of one-dimensional quasiperiodic systems formed by structural elements whose distribution is associated to a Sturmian word. The proposed method allows to relate a value of the generating parameter $\alpha \in [0, 1]$ with a system. By sweeping out the values of such generating parameter, we can form a family of structures with determined properties as for instance can be the dispersion relations or the distribution of resonances. Then we seek to relate this generating parameter α to the admitted frequencies in the system by means of the so–called *bulk spectrum*.

The transfer matrix method (TMM) is an analytical method suitable for the study of one–dimensional wave propagation. The TMM allows to express the state variables of the problem associated to a point of the system from those of another point by means of a product of matrices, connecting the system properties between both points. Denote by $\mathbf{u}(x, t)$ the state vector in time-domain at position x and at instant t. In general, this vector contains both node displacements and internal forces of the system. Considering harmonic motion, we can write $\mathbf{u}(x, t) = \mathbf{u}(x)\, e^{i\omega t}$. Let us consider two points in the system x_j and x_k and denote $\mathbf{u}_j = \mathbf{u}(x_j)$, $\mathbf{u}_k = \mathbf{u}(x_k)$. Then, the transfer matrix of the system between nodes j and k, such that $x_j < x_k$ is a square matrix $\mathbf{T}_{jk}$ such that

$$\mathbf{u}_k = \mathbf{T}_{jk}\, \mathbf{u}_j \; . \tag{7}$$

As we menthioned above, considering our system as Sturmian means that certain element parameter Θ is tunned following the Sturmian pattern $\mathcal{B}(\alpha)$ associated to certain number $\alpha = [0; a_1, \ldots, a_n]$. In the Fig. 2 we present a schematic sketch of a Sturmian system where the supercell given by $\mathcal{B}(\alpha)$ is periodically repeated.

Denoting $\mathbf{T}_j$ the transfer matrix between nodes $j-1$ and j, the relationship between the state vectors at the two ends of the unit cell can be written as

$$\mathbf{u}_N = (\mathbf{T}_N \cdot \cdots \cdot \mathbf{T}_1)\; \mathbf{u}_0 \equiv \boldsymbol{\mathcal{T}}(\alpha)\, \mathbf{u}_0 \; . \tag{8}$$

The above expression holds for any one-dimensional dynamic model, regardless of the algorithm used for its construction. In the Sturmian case each block emerges from concatenation of previous blocks according to the rule (2). If we consider the transfer matrix associated to the kth Sturmian block $\mathcal{B}_k$, $\boldsymbol{\mathcal{T}}_k$, we can stablish the recursion

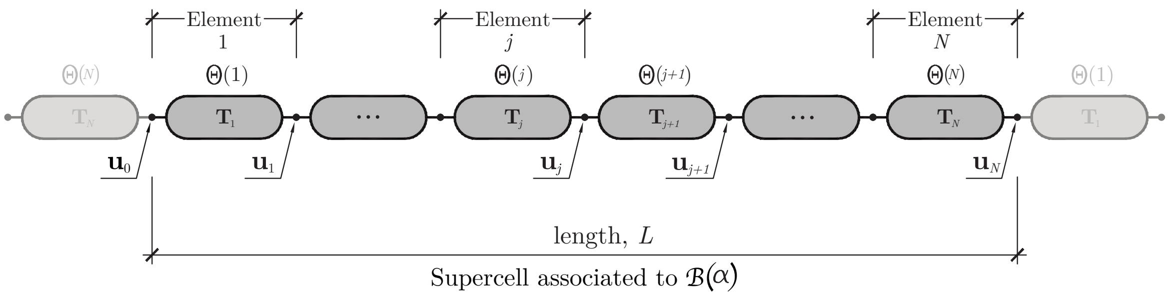

Figure 2: A one-dimensional Sturmian structured system associated to the number α. The binary parameter $\Theta(j) \in \{\theta_p, \theta_q\}$ changes its value according to the Sturmian pattern given by the block $\mathcal{B}(\alpha)$

$$\begin{aligned} \boldsymbol{\mathcal{T}}_k &= \boldsymbol{\mathcal{T}}_{k-2}\boldsymbol{\mathcal{T}}_{k-1}^{a_k} \, , \quad 1 \le k \le n \, , \\ \boldsymbol{\mathcal{T}}_{-1} &= \mathbf{T}_q \, , \quad \boldsymbol{\mathcal{T}}_0 = \mathbf{T}_p \, . \end{aligned} \tag{9}$$

The TM of the unit cell is then $\boldsymbol{\mathcal{T}}(\alpha) = \boldsymbol{\mathcal{T}}_n$ and relates the state variables $\mathbf{u}_0$ and $\mathbf{u}_N$ yielding

$$\mathbf{u}_N = \boldsymbol{\mathcal{T}}(\alpha)\, \mathbf{u}_0 \; . \tag{10}$$

Applying the Bloch theorem to the supercell we know that $\mathbf{u}_N = e^{\mathrm{i}\kappa L}\mathbf{u}_0$, thus Eq (10) can be written then as the linear eigenvalue problem

$$[\boldsymbol{\mathcal{T}}(\alpha) - \lambda\, \mathbf{I}]\; \mathbf{u}_0 = \mathbf{0} \, , \tag{11}$$

where the parameter is $\lambda = e^{\mathrm{i}\kappa L}$. As known, the TM depends on the frequency ω. If there exist real solution for the wavenumber κ from Eq. (11) the corresponding frequency ω is in the passband of the system and a wave of frequency ω is said to be admitted in the medium. On the contrary, if it is in a bandgap or stopband, wave cannot be transmitted, i.e., is evanescent with an exponentially decaying amplitude (complex wavenumber). In most structural models of rods and beams the transfer matrices are 2×2 or 4×4 in size. For them, closed forms for the dispersion relations can be derived.

If $\boldsymbol{\mathcal{T}}(\alpha)$ is a 2×2 matrix then the characteristic polynomial of Eq. (11) is

$$\det\left[\boldsymbol{\mathcal{T}}(\alpha) - \lambda\, \mathbf{I}\right] = \lambda^2 - \mathrm{tr}\left[\boldsymbol{\mathcal{T}}(\alpha)\right]\, \lambda + \det\left[\boldsymbol{\mathcal{T}}(\alpha)\right] \; , \tag{12}$$

where $\mathrm{tr}(\bullet)$ stands for the matrix trace operator. The fact that transfer matrix is unimodular [26] notably simplifies the problem resulting, after some straight operations, the final expression for 2×2 TM spectrum

$$\cos(\kappa L) = \frac{1}{2} \mathrm{tr}\left[\boldsymbol{\mathcal{T}}(\alpha)\right] \ . \tag{13}$$

This approach can be extended to consider system described under 4×4 transfer matrices [5] obtaining,

$$\cos(\kappa L) = \frac{1}{4}\left[\mathrm{tr}\left[\boldsymbol{\mathcal{T}}(\alpha)\right] \pm \sqrt{2\,\mathrm{tr}\left[\boldsymbol{\mathcal{T}}^2(\alpha)\right] - \mathrm{tr}^2\left[\boldsymbol{\mathcal{T}}(\alpha)\right] + 8}\right] \ . \tag{14}$$

The two solutions obtained lead to two dispersion branches related to waves of different nature in the model. Thus, for instance, in the case of Timoshenko beams, (an example of 4×4 TM), both solutions correspond to the spectrum of pure bending and shear waves associated with each frequency.

From both Eqs (13) and (14) the wavenumber $\kappa(\omega)$ can be expressed analytically as function of frequency [24]. Admitted frequencies are those values of ω which lead to a real wavenumber $\kappa(\omega)$. For 2×2 TM, this can be reduced to the condition $-2 \leq \mathrm{tr}\left[\boldsymbol{\mathcal{T}}(\alpha)\right] \leq 2$. For 4×4 TM bandgaps are defined as those frequencies which make the right hand side of Eq. (14) to be higher than 1 in absolute value. Collecting the admitted frequencies, they can be arranged along a line so that passbands are depicted as segments and stopbands are the bandgaps between them. Repeating the process for the whole range of α the passbands and stopbands forme a figure, called *bulk spectrum* (BS). The finer the discretization of the interval $[0, 1]$, the better resolution of this graphical figure, which allows to visualize at a single glance the quasiperiodic profile of our system based on the Sturmian sequences. In the following section we will present some properties of BS that can be established a priori and that will be tested later in the numerical examples.

5 Numerical Examples

In this section we are going to present two examples of systems following a Sturmian quasiperiodic pattern. We will study their spectral properties.

5.1 Example 1. Compressional waves in discrete systems

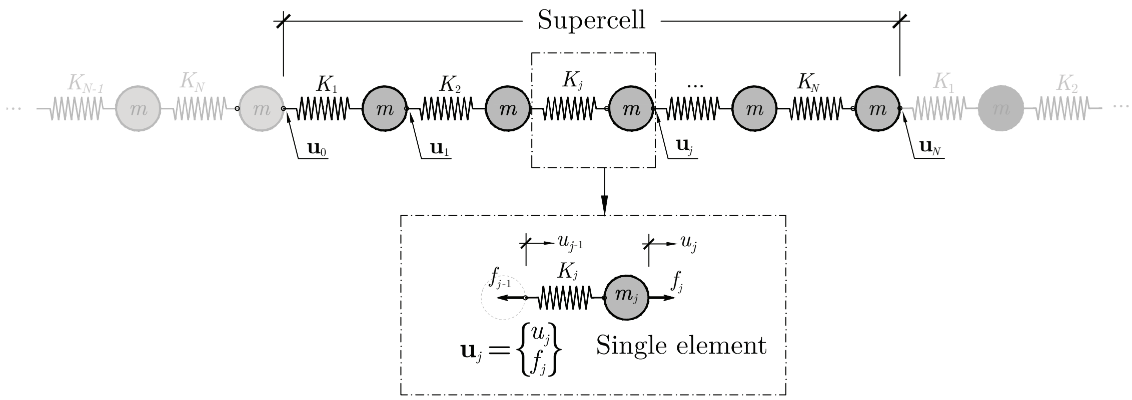

Figure 3: (Example 1) Sturmian quasiperiodic distribution of rigidities K_j in a discrete spring-mass system.

Lets consider a discrete spring-mass lattice (see Fig. 3). Following the methodology described above, the system consists of the periodic concatenation of N single elements formed by a mass and a linear spring, with parameters m_j and K_j, respectively. Here we are going to consider that the rigidities K_j are arranged along the chain following the Sturmian sequence associated to $\alpha = [0; a_1, \ldots, a_n] \in [0, 1]$.

The horizontal displacements in time domain are described by $\mathfrak{u}_j(t)$. We consider harmonic motion with circular frequency ω, i.e., $\mathfrak{u}_j(t) = u_j e^{\mathrm{i}\omega t}$ and the harmonic force acting on the link elements is denoted by $\mathfrak{f}_j(t) = f_j e^{\mathrm{i}\omega t}$. Then the state vector in the frequency domain can be defined as $\mathbf{u}_j = \{u_j, f_j\}^T$. As shown in Fig. 3, state vectors can be located at both ends of each element. The relationship between each state vector and the preceding one is given by the product of the respective transfer matrices associated to the mass and to the spring [26], i.e.

$$\mathbf{u}_j = \left\{ \begin{array}{c} u_j \\ f_j \end{array} \right\} = \left[\begin{array}{cc} 1 & 0 \\ -m_j\omega^2 & 1 \end{array} \right] \left[\begin{array}{cc} 1 & -\frac{1}{K_j} \\ 0 & 1 \end{array} \right] \left\{ \begin{array}{c} u_{j-1} \\ f_{j-1} \end{array} \right\}$$

$$= \left[\begin{array}{cc} 1 & \frac{1}{K_j} \\ -m_j\omega^2 & 1 - \frac{m_j\omega^2}{K_j} \end{array} \right] \left\{ \begin{array}{c} u_{j-1} \\ f_{j-1} \end{array} \right\} \equiv \mathbf{T}(m_j, K_j)\, \mathbf{u}_{j-1} \,, \quad (15)$$

where $\mathbf{T}(m_j, K_j)$ denotes the transfer matrix of the jth element. In this notation is highlighted the fact that m_j and K_j are the dynamical parameters. As we have explained previousy, in our example $\Theta(j) = K_j$, remaining constant the masses, i.e. $m_j = m$ for all j. The parameter K_j takes values from the binary set $\{K_p, K_q\}$ according to what is specified in the Sturmian block $\mathcal{B}(\alpha)$, which in turn results a binary word from the alphabet $\{p, q\}$. In Fig. (4) we present three systems associated with the numbers $\alpha = \{2/7, 1/2, 7/8\}$. We have determine the dispersion relations as well as the representation of the wave frequency bands (spectrum bands).

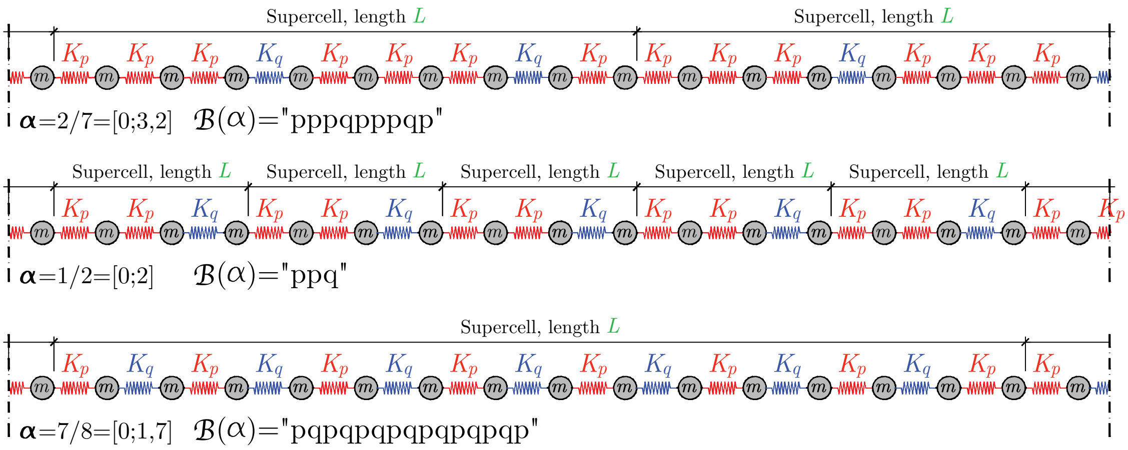

Figure 4: (Example 1) Three spring-mass systems associated to three numbers $\alpha = \{2/7,\ 1/2,\ 7/8\}$. The dispersion relation is determined assuming that the supercell of length L is distributed periodically

Using the particular values $K_p = 1$ N/m, $K_q = 2K_p = 2$ N/m, and $m = 1$ kg, the dispersion curves can be constructed sweeping out the range of frequencies $0 \leq \omega \leq 3$ rad/s and solving the equation:

$$\cos(\kappa L) = \frac{1}{2}\, \mathrm{tr}\, [\boldsymbol{\mathcal{T}}(\alpha)] \quad (16)$$

where κL is the dimensionless wavenumber and L stands for the supercell length. The fact that the supercell is made up of single elements with different parameters leads to heterogeneity and therefore to the appearance of passbands and stopbands. It turns out [27, 5] that the amount of passbands, coincides with the size of the Sturmian sequence which in turn is closely related to the associated number α by $N = \mathcal{N}(\alpha)$ (see [24]). Projection on a vertical line of the whole set of admitted frequencies leads to a simplified representation of passbands and stopbands, resulting a vertical line in pieces that can be associated to the number α, generator of the chain. The graphical representation that arises from the repetition of the process over the entire interval $0 \leq \alpha \leq 1$ gives rise to a figure like that of fig. 5. We call this representation the *Sturmian bulk spectrum*. As it can be observed, it has a fractal nature. All the details can be found in [24].

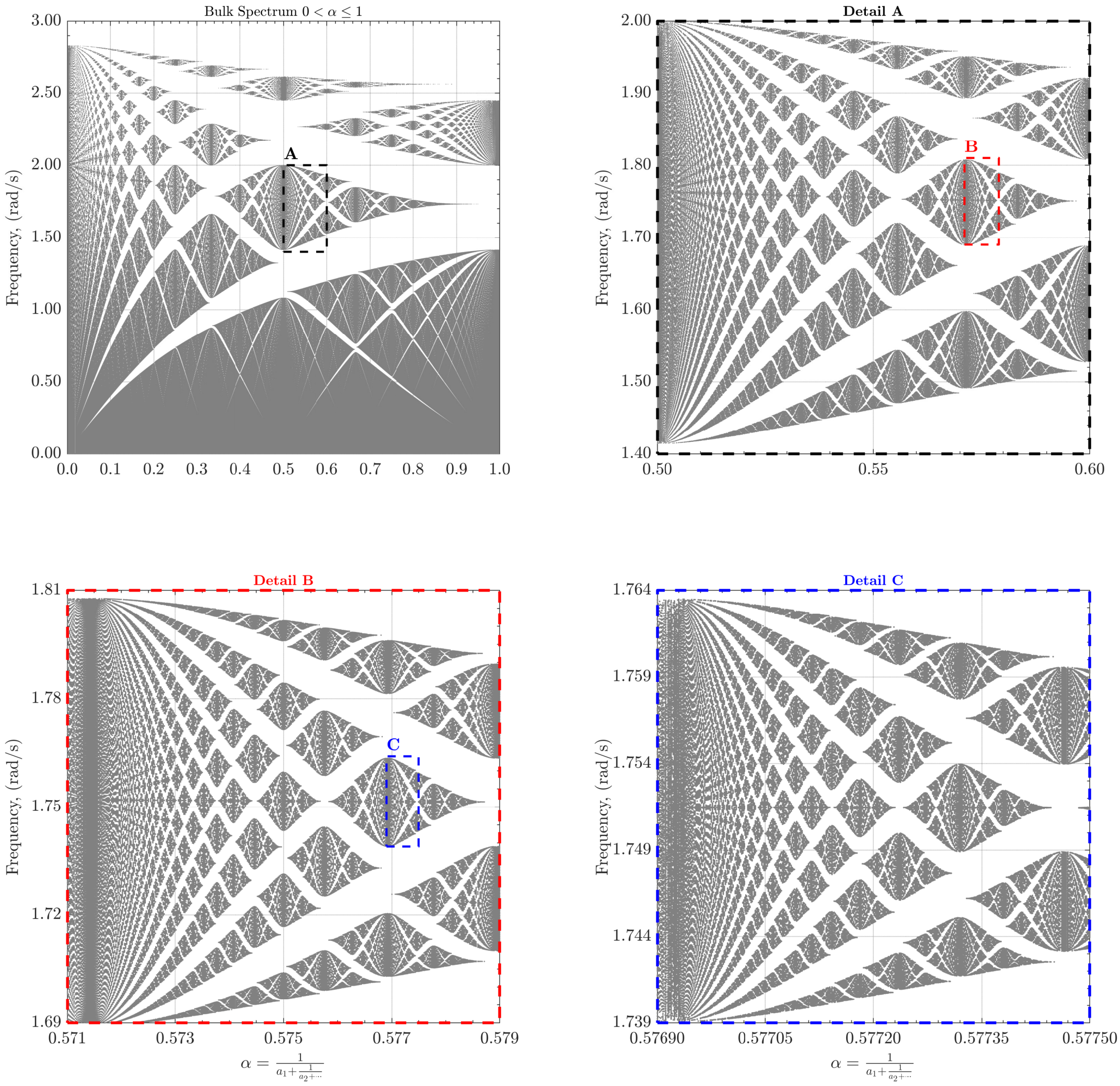

Figure 5: Sturmian bulk spectrum of a spring-mass system with quasiperiodic distribution of rigidities K_j. Top-left: bulk spectrum for the whole range of generator parameter $0 \leq \alpha \leq 1$. Top-right, bottom-right and bottom-left: details A, B and C to visualize the selfsimilar structure of the bulk spectrum.

5.2 Example 2. Compressional waves in rods

We assume an infinite medium formed by single elements of length l. The jth element has stiffness and mass properties given by EA_j and ρA_j, where EA_j and ρA_j stand for the compressional sectional stiffness and the mass per unit of length, respectively. In order to simplify the notation, the parameters EA_j and ρA_j are understood as the products of the Young modulus E_j and the density ρ_j and the area of the cross section A_j, associated to the jth element. As known, horizontal displacement $u(x,t)$ and internal force $f(x,t)$ in the jth element are related by

$$\frac{\partial u}{\partial x} = \frac{f(x,t)}{EA_j} \quad , \quad \frac{\partial f}{\partial x} = \rho A_j \frac{\partial^2 u}{\partial t^2} \; . \tag{17}$$

Assuming again harmonic motion with $u(x,t) = U(x)\, e^{\mathrm{i}\omega t}$ and $f(x,t) = F(x)\, e^{\mathrm{i}\omega t}$, Eqs. (17) yields

$$\left\{ \begin{array}{c} U'(x) \\ F'(x) \end{array} \right\} = \left[\begin{array}{cc} 0 & 1/EA_j \\ -\omega^2 \rho A_j & 0 \end{array} \right] \left\{ \begin{array}{c} U(x) \\ F(x) \end{array} \right\} \; , \tag{18}$$

where $(\bullet)' = d(\bullet)/dx$ denotes the space–domain derivative. The state vector $\mathbf{u}(x) = \{U(x), F(x)\}^T$ verifies then $\mathbf{u}' = \mathbf{W}_j(\omega)\, \mathbf{u}$, which integrating between $x = 0$ and $x = l$ give rise to the transfer matrix of a single element, $\mathbf{u}(l) = e^{\mathbf{W}_j(\omega) l}\, \mathbf{u}(0)$, where

$$\mathbf{T}_j(\omega) = e^{\mathbf{W}_j(\omega) l} = \left[\begin{array}{cc} \cos\mu_j & \frac{l}{EA_j\, \mu_j} \sin\mu_j \\ -\mu_j \frac{EA_j}{l} \sin\mu_j & \cos\mu_j \end{array} \right] \quad , \quad \mu_j = \omega\, l \sqrt{\frac{\rho A_j}{EA_j}} \; . \tag{19}$$

In this example, Θ is the sectional stiffness EA, i.e., $EA_j \in \{EA_p, EA_q\}$.

Fig. 6 presents the bulk spectra for the ratio $EA_p/EA_q = 4$. Other values of this ratio can be found in [24]. We know that $\alpha = 0$ corresponds to the continuous homogeneous rod with parameters EA_q and ρA_q with no stopbands in the whole frequency band. On the other side, $\alpha = 1$ gives rise to the periodic binary system "$pqpqpq\ldots$". Since rods are continuous structures, we will find passbands in the whole frequency band. However the general pattern of bands distribution strongly depends on the contrast between both EA_p and EA_q. Furthemore, a periodicity is observed in the vertical direction (frequency axis) of the bulk spectrum (see [24]).

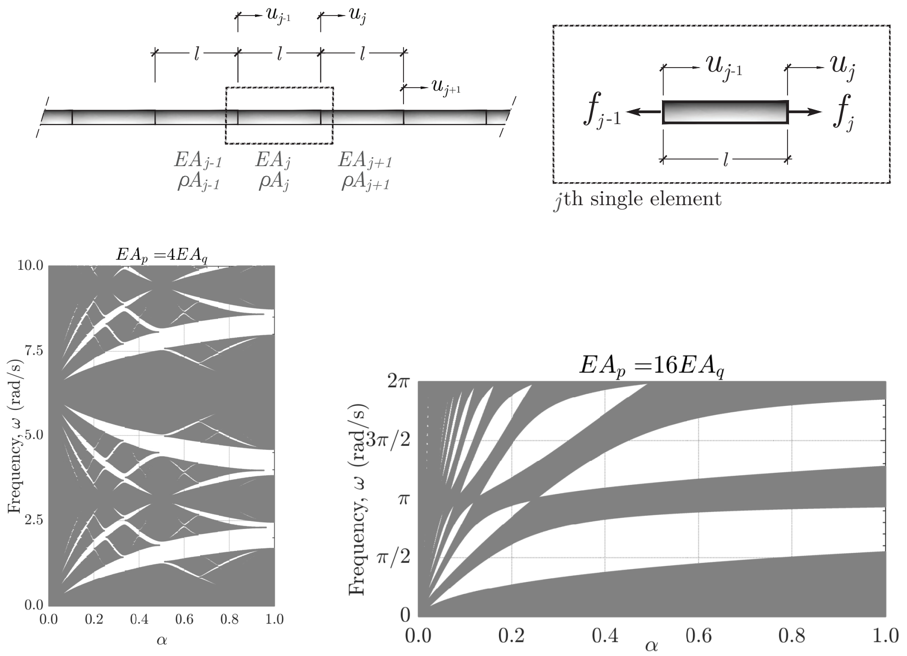

Figure 6: Upper part: Details of the horizontal displacement $u(x,t)$ and internal forces $f(x,t)$ in the jth element. Lower part: Sturmian bulk spectra of a rod (compressional waves) with quasiperiodic variation of elastic sectional stiffness between values $\{EA_p, EA_q\}$: The bottom left plot shows the bulk spectrum for tthe ratio $EA_p = 4EA_q$. Darkened regions show frequency passbands. The plot on bottom right represents the frequency passbands obtained from the analytical expression (23), for the value of EA_p/EA_q considered

Both, the bands width and the periodicity can be explained and somehow quantified studying the spectrum of the systems associated to the numbers given by the sequence $\{\alpha_r = 1/r\}_{r=1}^{\infty}$. The associated Sturmian block is $\mathcal{B}(\alpha_r) = p \overset{r}{\ldots} p\, q$ and therefore the transfer matrix yields

$$\boldsymbol{\mathcal{T}}(\alpha_r) = \mathbf{T}_q(\omega)\, \mathbf{T}_p^r(\omega) \,, \tag{20}$$

where

$$\begin{aligned} \mathbf{T}_q(\omega) &= e^{\mathbf{W}_q(\omega) l} = \begin{bmatrix} \cos\mu_q & \frac{l}{EA_q\,\mu_q}\sin\mu_q \\ -\mu_q \frac{EA_q}{l}\,\sin\mu_q & \cos\mu_q \end{bmatrix} \quad , \quad \mu_q = \omega\, l\sqrt{\frac{\rho A_q}{EA_q}} \,, \\ \mathbf{T}_q^r(\omega) &= e^{\mathbf{W}_p(\omega)(rl)} = \begin{bmatrix} \cos(r\mu_p) & \frac{l}{EA_p\,\mu_p}\sin(r\mu_p) \\ -\mu_p \frac{EA_p}{l}\,\sin(r\mu_p) & \cos(r\mu_p) \end{bmatrix} \quad , \quad \mu_p = \omega\, l\sqrt{\frac{\rho A_p}{EA_p}} \,. \end{aligned} \tag{21}$$

Admissibles frequencies correspond to the values of $\omega \in \mathbb{R}$ such that $-1 \le z_r(\omega) \le 1$, where $z_r(\omega)$ stands for the half trace of the transfer matrix, which can be expressed as

$$\begin{aligned} z_r(\omega) &= \frac{1}{2}\,\mathrm{tr}\left[\mathbf{T}_q(\omega)\mathbf{T}_p^r(\omega)\right] \\ &= \frac{(1+\lambda)^2}{4\lambda}\cos\left(\frac{\lambda + r}{\lambda}\,\frac{\omega}{lc_q}\right) - \frac{(1-\lambda)^2}{4\lambda}\cos\left(\frac{\lambda - r}{\lambda}\,\frac{\omega}{lc_q}\right) \,, \quad \lambda = \sqrt{\frac{EA_p}{EA_q}} \,,\ c_q = \sqrt{\frac{EA_q}{\rho A_q}} \end{aligned} \tag{22}$$

The conditions for the above expression to be periodic in frequency is that $(\lambda + r)/(\lambda - r)$ is rational, something that it holds provided that λ is rational. In fig. 6 the bulk spectrum for $\lambda = 2$ has been plotted. Along the frequency direction, the figure has a periodicity equal to $\Delta\omega = \pi\lambda c_q/l$. The particular values of the parameters are $\rho A_p = \rho A_q = 1$ kg/m, $EA_p = \lambda^2 EA_q$, $EA_q = 1$ N/m, $c_q = 1$ m/s. Therefore, $\Delta\omega = 2\pi$ rad/s. The plot shows clearly the periodicity not only for those values corresponding to $\alpha_r = 1/r$, but also for the whole range $0 \le \alpha \le 1$. The higher the ratio $\lambda = \sqrt{EA_p/EA_q}$, the more contrast between both rigidities. It is then expected that the passbands become narrower, as indeed occurs (see [24]).

Finally, we present a formula of the spectrum depending analytically on α and ω. Previous formula does not include the number α explicitly. We wonder how the bands are distributed if we do $r = 1/\alpha$, allowing α to take any real number in the range $0 \le \alpha \le 1$, leading to the new formula

$$Z(\alpha,\omega) = \frac{(1+\lambda)^2}{4\lambda}\cos\left(\frac{\alpha\,\lambda + 1}{\alpha\,\lambda}\,\frac{\omega}{lc_q}\right) - \frac{(1-\lambda)^2}{4\lambda}\cos\left(\frac{\alpha\,\lambda - 1}{\alpha\,\lambda}\,\frac{\omega}{lc_q}\right) \tag{23}$$

It is important to note that, although it is a closed form, results to be an expression derived after substituting α_r by α. Its representation in figs. 6 (bottom right plot) is made in order to numerically observe how passbands and stopbands are preproduce. Thus, we can say that

- The representation of the set $\{(\alpha,\omega) : -1 \le Z(\alpha,\omega) \le 1,\}$ reproduces the global form of the wider stopbands of the original bulk spectrum, but it does for $0 \le \omega \le \Delta\omega = \pi\lambda c_q/2l$. Further, the form is completely different. In the fig. 6, it has only been depicted this range, which covers the half period of the bulk spectrum.
- Eq. (23) reproduces the width pattern of the passbands: the larger the ratio λ, the narrower the passbands (see [24]).
- The fractal structure of the spectrum is not replicated. The admitted frequency bands do not show self-similarity.

Research on the formula developed and the explanation of the different phenomena observed is left for future work.

6 Conclusions

In this paper we have studied the dynamical properties of heterogeneous elastic structured media with quasiperiodic pattern. These quasiperiodic patterns have been obtained through the use of Sturmian sequences where, considering any real number in the interval [0,1] given as a continued fraction, alllows us to construct a word or sequence from a binary alphabet. We can relate this with two different values of one single parameter from a mechanical system. Dynamical properties and dispersion relations of Sturmian mechanical systems have been analytically determined using the transfer matrix method. This method allow us to study the self-similarity of the bulk spectrum. These properties have been validated and visualized along two numerical examples. In the first one a spring-mass system has been analized where the spring constants have been used as the quasiperiodic parameter. In the second one a rod is considered with the sectional stiffness being changed according to the Sturmian pattern. In this case an analytical formula has been derived that partially reproduces the pattern of the passbands for low frequencies. In both cases, the complete bulk spectrum of admitted states or frequencies of the system have been obtained and the results derived from the theoretical analysis validated.

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