Fitting a measured spectrum from a nonlinear structure to produce a model

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Abstract

One method of performing an experimental modal analysis is to apply an impulse to the structure and measure the vibration decay. If the structure is nonlinear this decay contains much useful information about the system such as instantaneous natural frequencies and damping. A method for extracting such nonlinear properties is described in this paper. It is shown that an appropriate general model can be obtained by fitting a Laurent series to each mode in the spectrum of the decaying time history. The approach is to split the time history into nonlinear modes and then analyze the individual modes in both the time domain and the frequency domain. The starting point is to use narrow band filtering to split the time history into individual modes. Subsequent, analysis uses curve fitting in the time domain to remove noise. The output is the instantaneous frequency and damping for the structure under consideration.

1 Introduction

Experimental modal analysis is a well-developed procedure for determining the dynamic properties of a system. Details may be found in Ewins [1]. However, these methods are generally restricted to linear structures. This paper extends one of the standard methods of experimental modal analysis to nonlinear structures.

The starting point is to measure the free vibration decay of the structure following an impulse excitation. This is a standard procedure in experimental modal analysis. The decaying vibration time history typically contains a sum of the response from several modes all of which may be nonlinear. The first problem is thus to split the time history into several time histories each one containing the vibration decay of one mode. This is the substantial problem examined in this paper. Once this separation into modes has been achieved then a second analysis can be performed on each decaying mode to determine the instantaneous natural frequency and damping ratio.

The concept of a mode is well defined for linear structures where the linearity enables the superposition principle to be applied. Each mode is associated with a degree of freedom of the structure. Further, through the use of an eigensystem analysis it can be shown that the modes are uncoupled in mass and stiffness and generally only weakly coupled by damping. In particular, in free vibration a linear structure vibrates with its natural frequencies with each mode exhibiting an exponentially decaying amplitude controlled by damping. See Chapter 2 of Ewins [1] for full details. For a nonlinear structure none of these nice linear properties may apply. However, Rosenberg [2] suggest that modes will occur in nonlinear system. The methods of this paper also assume that the free response of a nonlinear structure can be expressed as a sum of decaying modes. If modal coupling is present, due to nonlinearity, then examination of the modes should reveal this coupling. No other assumptions are made, and it is expected that the time history of each mode will have changing natural frequencies and damping as the mode decays.

The use of an impulse excitation is considered here. The alternative is to use a shaker and apply a force to the structure. This can be advantageous where large forces are needed and by testing at various force levels nonlinear features such as natural frequencies changing with amplitude can be revealed. However, it is difficult to extract damping data from such testing. The advantage of free vibration following an impulse is that the structure exhibits its behaviour without interference and in particular changing frequencies and damping with amplitude occur naturally.

The paper is structured by first examining an overview of the method with the emphasis on how to separate the decaying response of each nonlinear mode from the overall response of the system. Subsequent sections then describe the details of the method. Extensive use is made of signal processing methods using both the time domain and the frequency domain.

One example is used throughout the paper. This provides consistency but the methods described are designed to be widely applicable and should be appropriate for nearly all nonlinear systems.

2 Overview of approach

An example of a time history obtained using impulse excitation is shown in Figure 1. This time history is the vibration decay of a beam-like structure with bolted joints (for details see Goyder[3]). The time history consists of the response of several of the nonlinear modes of the structure.

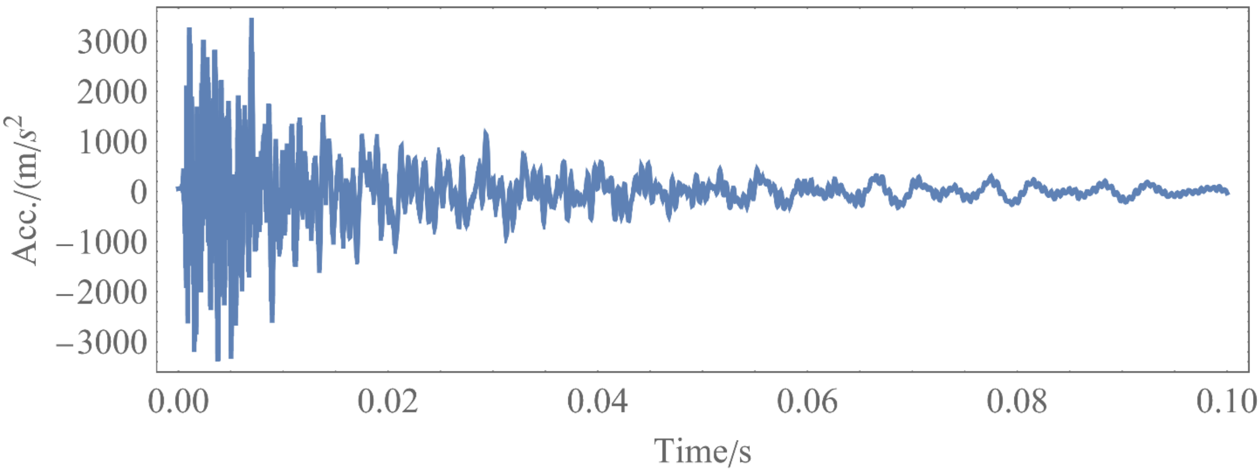

Figure 1. A measured acceleration time history

Figure 2 shows the modulus of the Fourier transform of the time history. Several peaks can be seen in the spectra each of which has a narrow bandwidth. These peaks correspond to the individual modes of the structure and the first 7 are labeled. There are other modes in the spectrum which are not labeled, and which will not be analyzed. The phase is measured but not shown.

Figure 3 shows an expanded range of the spectrum in Figure 2 covering the first two modes. The shape of the peaks departs significantly from that found in the spectrum of a linear structure. The vertical lines in Figure 3 identify narrow bands that contain most of the individual peaks.

The result of the signal processing procedures described here is to separate out the modes that compose the time history shown in Figure 1. These modes are shown in Figure 4 as time histories. The 7 plots correspond to the 7 peaks labeled in Figure 2. Here, we can see decay time histories that are straightforward to analyze. For example, mode 1 shows a rapid initial decay with large damping followed by a slow decay with small damping. The frequency of vibration also changes during the decay increasing for large times. Mode 2 is shown to be two close natural frequencies with nonlinear behaviour. Subsequent modes show further nonlinear behaviour. Each of these time histories can be formed into a nonlinear model for the mode.

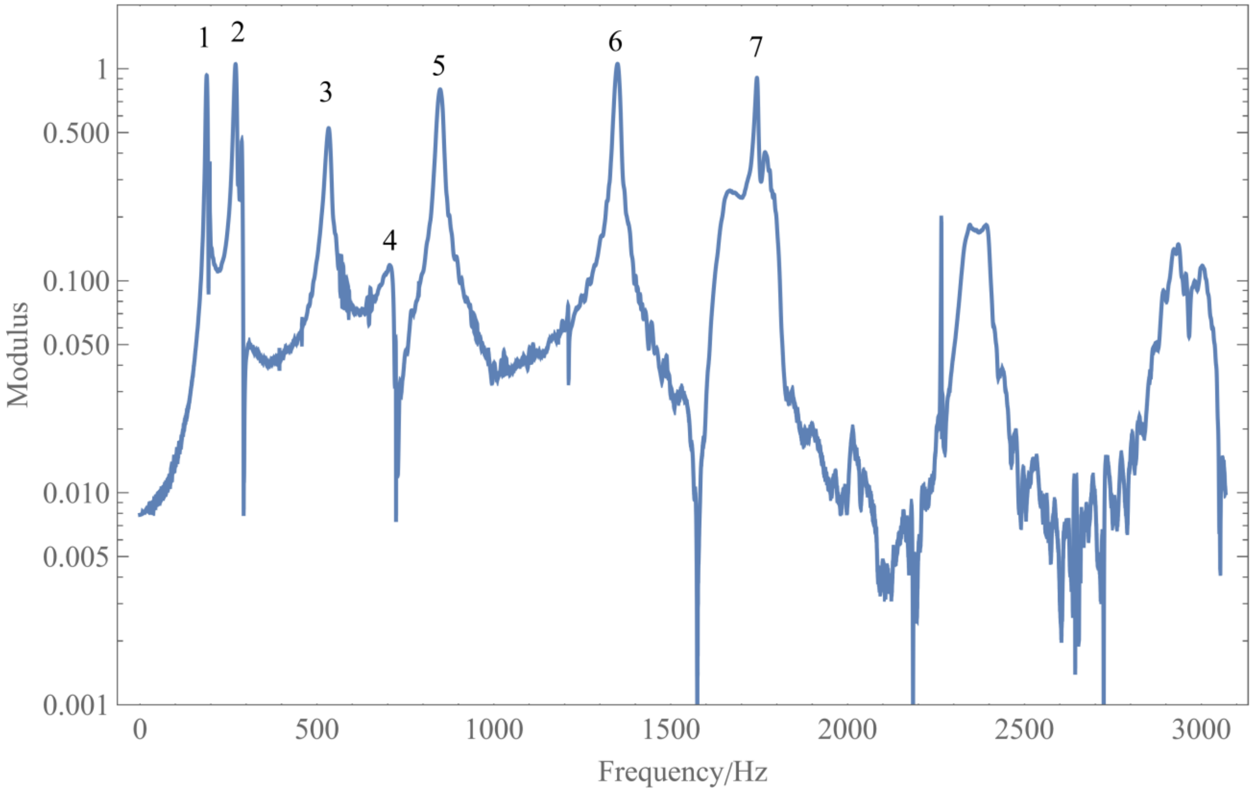

Figure 2. The Fourier transform of the time history in Figure 1. Seven peaks are labeled, and these modes will be extracted from the data.

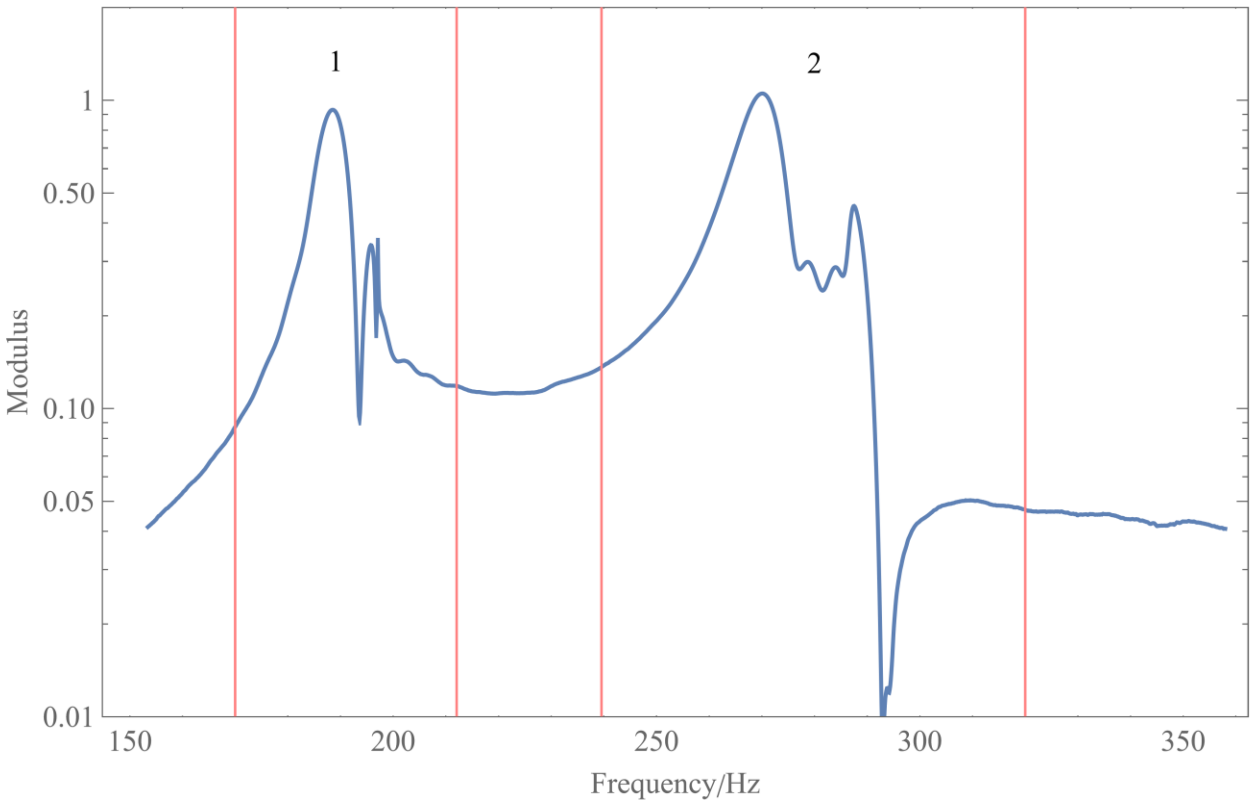

Figure 3. Expanded version of Figure 2 showing the first 2 modes. The vertical lines define intervals containing the peaks associated with mode 1 and mode 2.

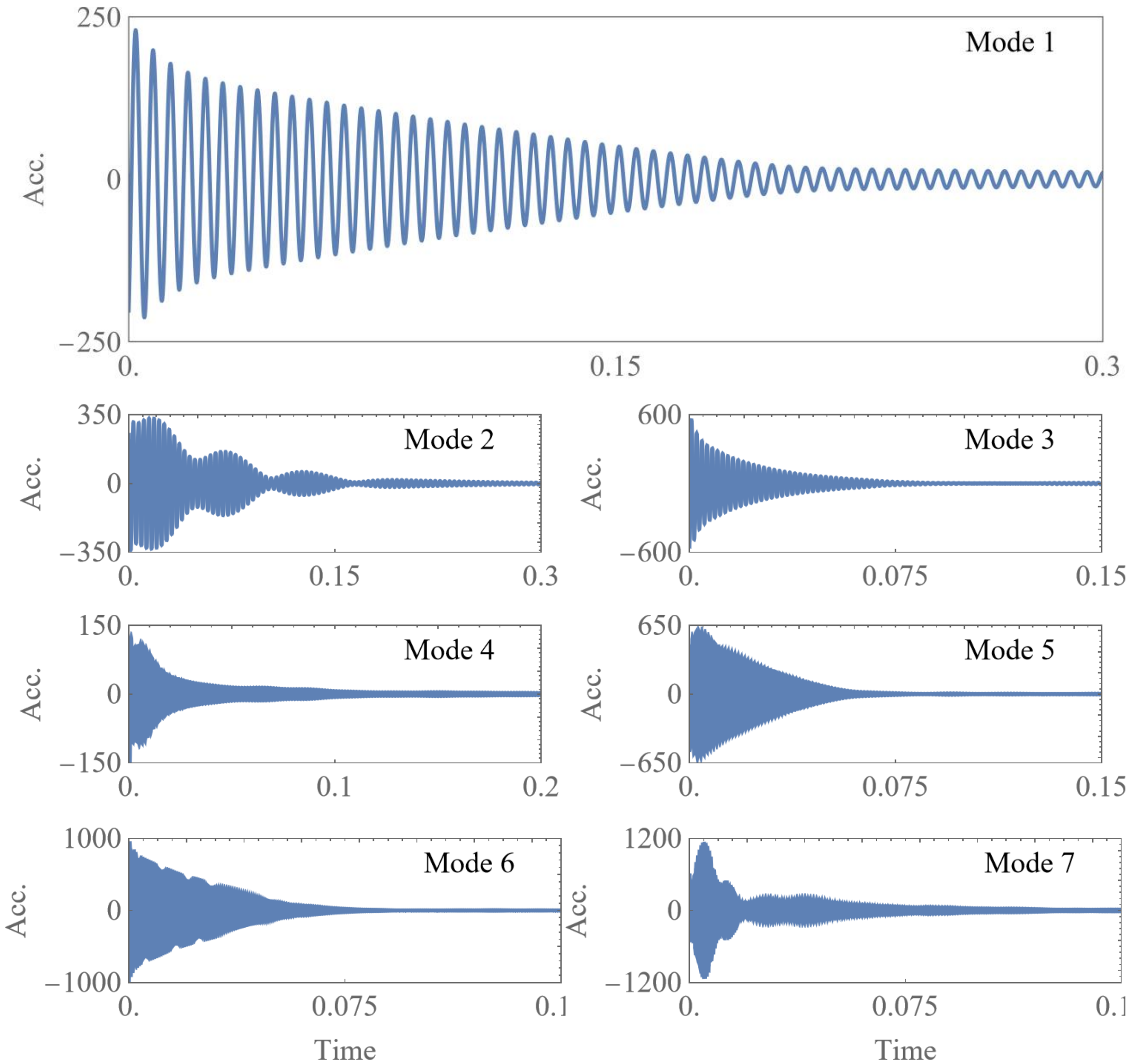

Figure 4. The first seven modes separated out from the time history in Figure 1.

Figure 5 repeats the spectrum shown in Figure 2 but adds the Fourier transforms of the individual time histories shown in Figure 4. The individual modes overlap and dealing with this overlap is a key issue in the method developed for extracting the individual modes.

Finally Figure 6 again shows the spectrum of Figure 2 but now the modeled modes shown in Figure 5 are subtracted from the measured data to show the error in the modelling process. It can be seen that the error is two magnitudes less than the measured spectrum and is flat with very little structure. This demonstrates that the model fitted to the measured data is highly accurate.

The methods used to extract the modes from the time history are discussed in the subsequent sections. Use is made of the Laurent series in modelling each mode. This is described next.

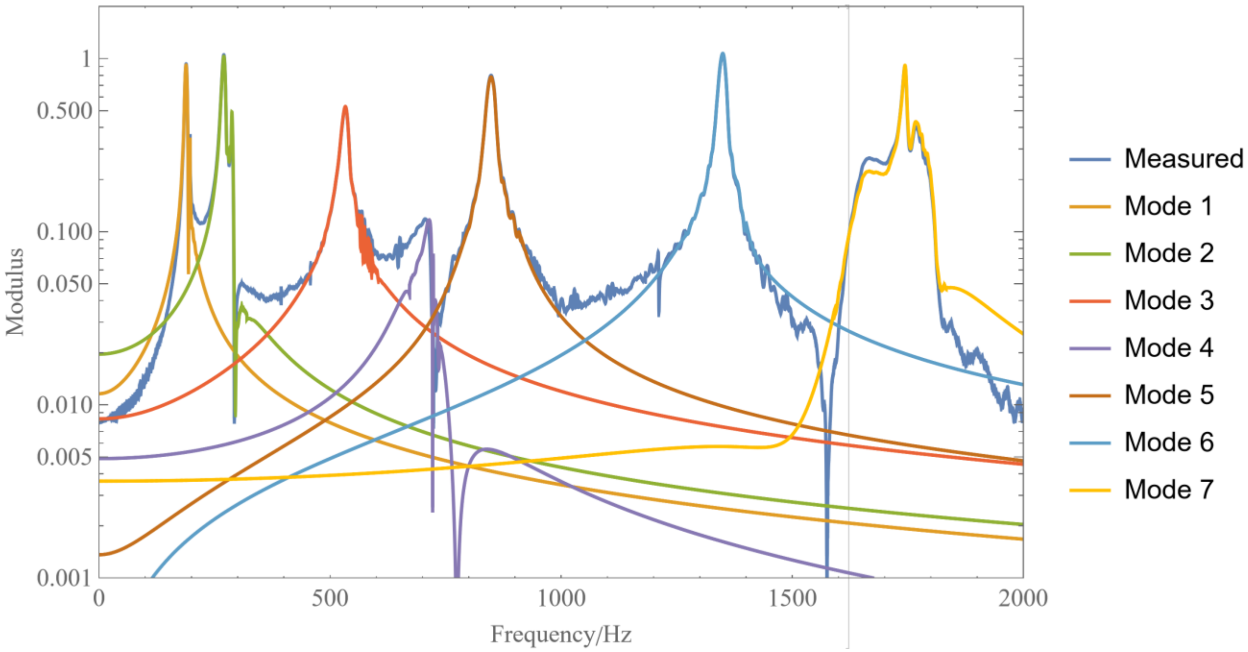

Figure 5. The measured spectrum with the seven extracted modes superimposed.

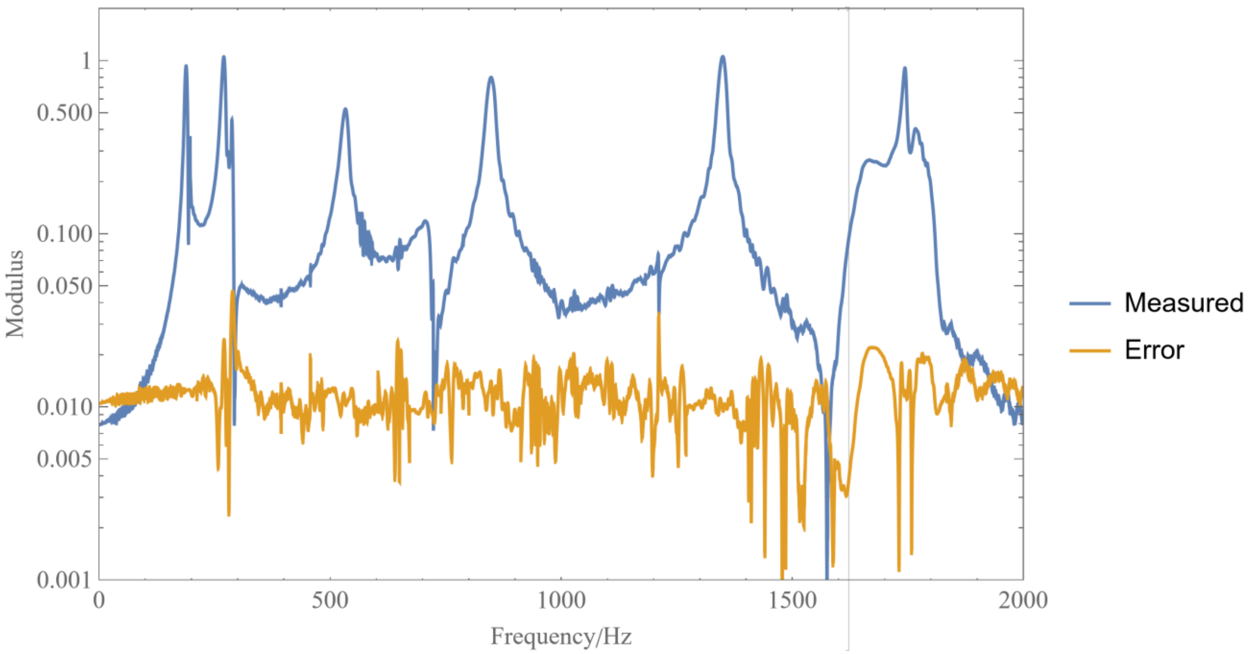

Figure 6. The measured spectrum with the seven modes subtracted showing the error.

3 Frequency domain model of nonlinear behaviour

In order to extract the modes from the time history it is necessary to have a model for the data. A frequency domain model is used. In conjunction with Figure 5 it was noted that the modes overlap in the frequency domain. It is this overlap that needs to be modeled so that modes can be separated. The model is based on the Laurent series which is described next.

The Fourier transform of a time history gives a spectrum. The spectrum is a representation of the time history in the frequency domain. A more general transform is the Laplace transform which again starts with the time domain but now maps the time domain to the entire complex plane. If a numerical Laplace transform is calculated, it is only possible to determine values on the right half plane. The values along the imaginary axis of the Laplace plane coincide with the Fourier transform. The right half plane is free from singularities but the un-calculable left half plane contains the singularities which define the nonlinear system. Each mode is assumed to contribute one singularity. Although it is not possible to determine the singularities on the left half plane it is possible to approximate their effect on a domain outside the location of the singularities. This is done by means of a Laurent series. The Laurent series for one singularity takes the form

$$H_{\boldsymbol{m}}(s) = \sum_{n=1}^{\infty} \frac{a_{m\,n}}{(s - s_m)^n} \tag{1}$$

where $H_m(s)$ is the Laplace transform of the time history of mode m, s is the Laplace variable, a_{mn} a coefficient and s_m a point on the complex plane which is the center about which the series is formed. See any good textbook on complex analysis for details of the Laurent series. It should be noted that s_m can be chosen as convenient. A good choice will result in the need for relatively few terms while a poor choice will require many terms. Since data is available along the imaginary axis this series can, in principle, be fitted to this data and a model for the right half of the complex plane obtained.

A further observation is that whatever the singularity being modeled there is always a companion singularity. Any singularity in the upper left hand plane is matched by a conjugate singularity on the lower half of the left hand plane. This is necessary so that the inverse Laplace transform gives rise to real time histories. Thus, a more complete model of one mode is given by the sum of the contributions from each singularity and its conjugate companion. The series expansion for each mode may written

$$H_{\boldsymbol{m}}(s) = \sum_{n=1}^{\infty} \left(\frac{a_{m\,n}}{(s - s_m)^n} + \frac{\overline{a_{m\,n}}}{(s - \overline{s_m})^n} \right) \tag{2}$$

where the over bar indicates a complex conjugate. If the n = 1 term is written out in full and the centre of expansion chosen as $s_m = -\zeta\,\omega_m + i\,\omega_m \sqrt{1-\zeta^2}$ then Equation 2 may be written as

$$H_{\boldsymbol{m}}(s) = \frac{2\left(\zeta\,\omega_{\boldsymbol{m}}\,a_r - a_i\sqrt{1-\zeta^2}\;\omega_{\boldsymbol{m}} + a_r\,s\right)}{s^2 + 2\,s\,\zeta\,\omega_{\boldsymbol{m}} + \omega_{\boldsymbol{m}}^{\boldsymbol{2}}} + \sum_{n=2}^{\infty} \left(\frac{a_{m\,n}}{(s - s_m)^n} + \frac{\overline{a_{m\,n}}}{(s - \overline{s_m})^n} \right) \tag{3}$$

where $a_{m\,1} = a_r + i\,a_i$. Thus, the first term is the familiar form for the linear single degree of freedom system where ω_m. is the natural frequency and ζ the damping ratio. The concept of fitting the Laurent series to nonlinear data is therefore a natural extension to the well developed problem of fitting a single degree of freedom system to measured spectra.

When working with the Laurent series and values from a Fourier transform the values of s correspond to the values on the imaginary axis and thus $s = i\,\omega$. For values of s that are distant from s_m, that is frequencies that are large or small compared to the centre of expansion, the denominator terms are large. Consequently, for this case only the first few terms will be relevant in the Laurent series the remaining higher order terms being negligible. This fact will be used extensively.

Unfortunately, the Laurent series is not convenient in the form given. When many terms in the series are needed the high power in the denominator leads to numerical instability. Nevertheless, the Laurent series is a completely general form for a nonlinear system in the frequency domain.

The next sections will show how the Laurent series can be used for separating the nonlinear modes from each other in the frequency domain.

4 Step 1: Reverse filtering and truncation

The objective of reverse filtering and truncating is twofold. First, to gain an initial understanding of each mode and second to determine the center of expansion for fitting a Laurent series.

The procedure is to start in the frequency domain and for each mode of interest identify a lower and upper frequency that contains the peak. These frequencies are then used to construct a narrow band digital filter.

The vertical lines in Figure 3 are examples of a frequency interval around a peak. This interval is referred to as the pass band of the filter.

When filtering the data, the original time history, for example Figure 1, is first reversed in time and then filtered. The data is then reversed again to reset the time. The reason for reversing the data is that the filter is a system with poles and zeros and will oscillate if suddenly excited. If the data is put through the filter in the normal forward condition the sudden start will cause filter oscillations that mix with the oscillations in the data. By reversing the data, the signal to be filtered rises slowly and no oscillations occur in the filter. Although filtering is performed by digital filters these filters still have responses that oscillate when subjected to a sudden change in input. This is particularly true for the type of narrow band filter being used here.

The issue with truncation is concerned with the time at which the reverse filtering is stopped. The condition required is for the filtering to be stopped when the reversed data finishes. If the filtering is continued, then the filter output will generate a further time history even though there is no input. This additional output is the energy in the filter decaying.

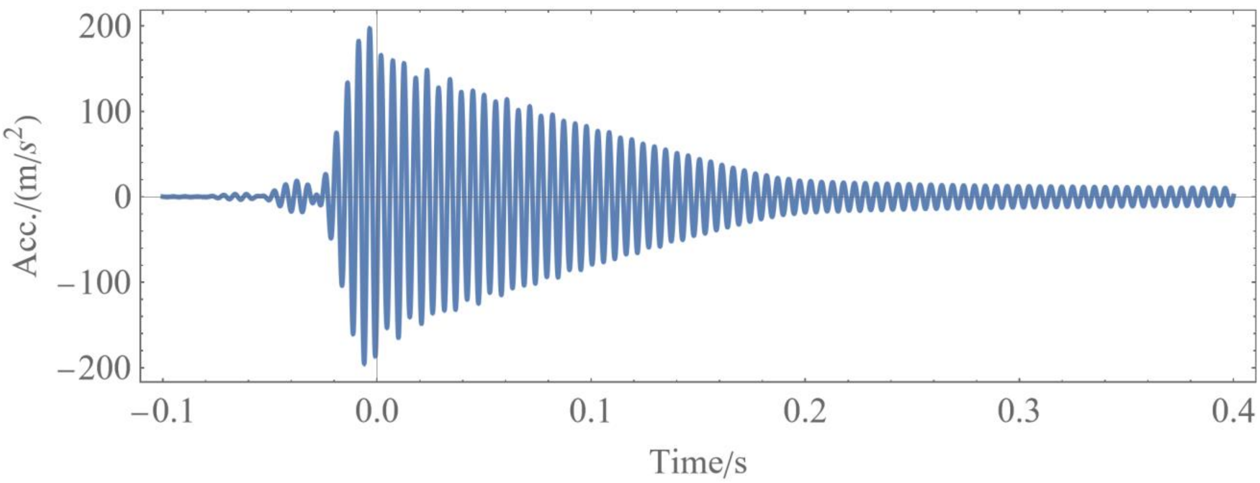

Figure 7. The measured time history after reverse filtering for the first mode.

Figure 7 shows the filtered data for the first mode. Note that the reverse filtering has generated a response that extends into negative time. For positive times the filtered data clearly shows the nonlinearity with a steeply decaying envelope followed by a slowly decaying envelope after about 0.2 seconds. This preliminary look at the data shows the nonlinearity being investigated. It can be shown, Goyder [4], that if the system is linear then the frequency and decay rate of the filtered data are exactly that of the system. However, if the system is nonlinear some further adjustment of the data is required so that instantaneous natural frequencies and damping ratios can be extracted.

Two versions of the Fourier transform of the filtered data shown in Figure 7 are shown in Figures 8. The Fourier transform of the measured data, the Fourier transform that includes the decay from the filter in

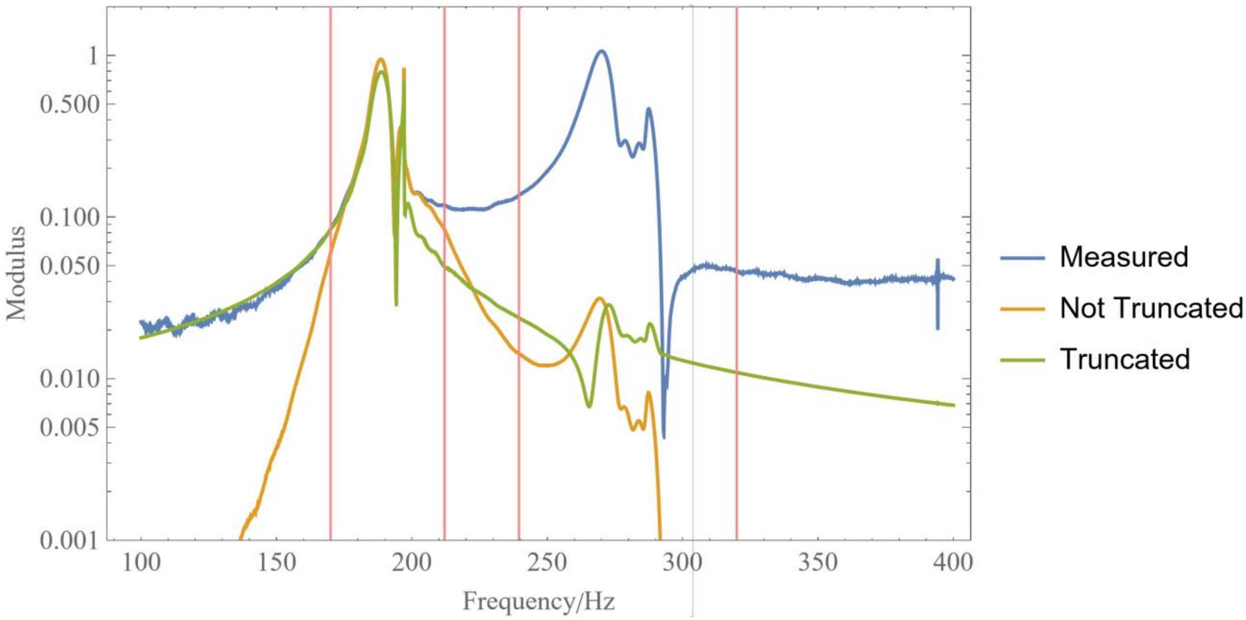

Figure 8. The measured spectrum and two spectra after reverse filtering the data. The effect of truncating the filtered data is illustrated.

negative time and the Fourier transform of the data that has been truncated are shown. Both of the spectra that have been filtered show how the neighboring modes have been suppressed by filtering leaving a much reduced content in the neighboring pass band. For the case where the data is truncated the spectrum is beneath the unfiltered data in the pass band with low and high frequency tails outside the filter pass band. For the case of the untruncated data the filtered spectrum lies on top of the unfiltered spectrum and the tails drop off very sharply outside the filter frequency pass band. The latter behaviour is as expected from a bandpass filter. However, the former spectrum is required for the signal processing procedure required here.

Most importantly, in the frequency domain, the tails of the truncated data outside the filter pass band coincide with the first few terms of the Laurent series with the higher order terms being negligible. Thus, by curve fitting the data outside the pass band an estimate may be made of the centre of expansion of the Laurent series.

This first step in signal processing is completed by determining the center of expansion for each mode of interest by curve fitting the tails of the filtered data with a few terms of the Laurent series; here three terms were used.

5 Step 2: Separation of modes

In this step the individual modes are separated using the data of the unfiltered spectrum. If each mode of the unfiltered spectrum is examined, see Figure 5, then within each pass band surrounding a peak there is measured data to which there are additional contributions from the tails of the neighboring modes. Between the pass bands there are contributions from the tails of all the modes. The approach taken is to curve fit the data between the pass bands with a few terms of the Laurent series from each mode. From the previous step the centers of expansion have been identified so that only the coefficients a_{mn} need to be determined. This is a linear least squares curve fitting calculation so easily performed. Only a few terms in the series for each mode need be used, in the case being described three terms were used.

Once a few terms in the Laurent series have been determined for each mode it is straightforward to separate the modes. Within each pass band the tails of the neighbouring modes are subtracted leaving measured data for the mode of interest. Outside the pass band the spectrum is modelled by the terms derived from curve fitting. The complete spectrum for each mode from zero frequency to the maximum measured is thus made up from the fitted data outside the pass band and measured data within the pass band.

The separated modes are shown in Figure 4 in the time domain and in Figure 5 in the frequency domain.

6 Step 3: Modelling of individual modes

The final step is to model the individual modes. This is done in the time domain. The spectrum from each mode is inverse Fourier transformed to produce a time history. This time history will have measurement noise which needs to be removed. This involves further curve fitting.

The model used for the time domain is that of a linear differential equation with time dependent coefficients. The equation used is

$$\ddot{y} + \beta(t)\,\dot{y} + \alpha(t)^2\,y = 0 \tag{4}$$

where y is the acceleration (or velocity or displacement) α is the instantaneous frequency and β is the instantaneous damping.

The curve fitting is performed by rewriting the differential equation as a difference equation. Highly accurate estimates of the derivatives can be made by using centered differencing of high order, here 6th order. This results in a linear least squares curve fitting procedure which is straightforward to implement. The values of the fitted instantanious frequency and damping for the first mode are given in Figure 9.

As an alternative to instantaneous frequency and damping the results may also be given as a restoring force surface where the stiffness and damping are a function of amplitude.

7 Overall error

It is important to establish the errors involved in the above procedure. The final model is given in the time domain with values of instantaneous frequency and damping for each mode. How can the model be assessed for its accuracy?

In general, if a model determined from measured data is accurate there are two requirements. First, the difference between the data and the model, sometimes called the residual, should be small. Second, the residual data should have no structure since all the structure in the data should be included in the model. Determining if these requirements are satisfied by the models generated here is best performed in the frequency domain. Figure xx shows the result of subtracting the model data from the measured data. It can be seen that the residual is small, compared to the measured data. Further, the residual has a flat spectrum suggesting that all the structure has been removed. There are some exceptions around xx Hz where some structure in the residual is present. This may need more attention.

One advantage of looking at the errors in the frequency domain is that typically only a few modes are of interest. Consequently, the residual outside the frequencies of interest may be large. In the example data there are other modes at high frequencies that have not been modelled. If the data were examined in the time domain, these high frequency errors would make the success of the model fitting process unclear.

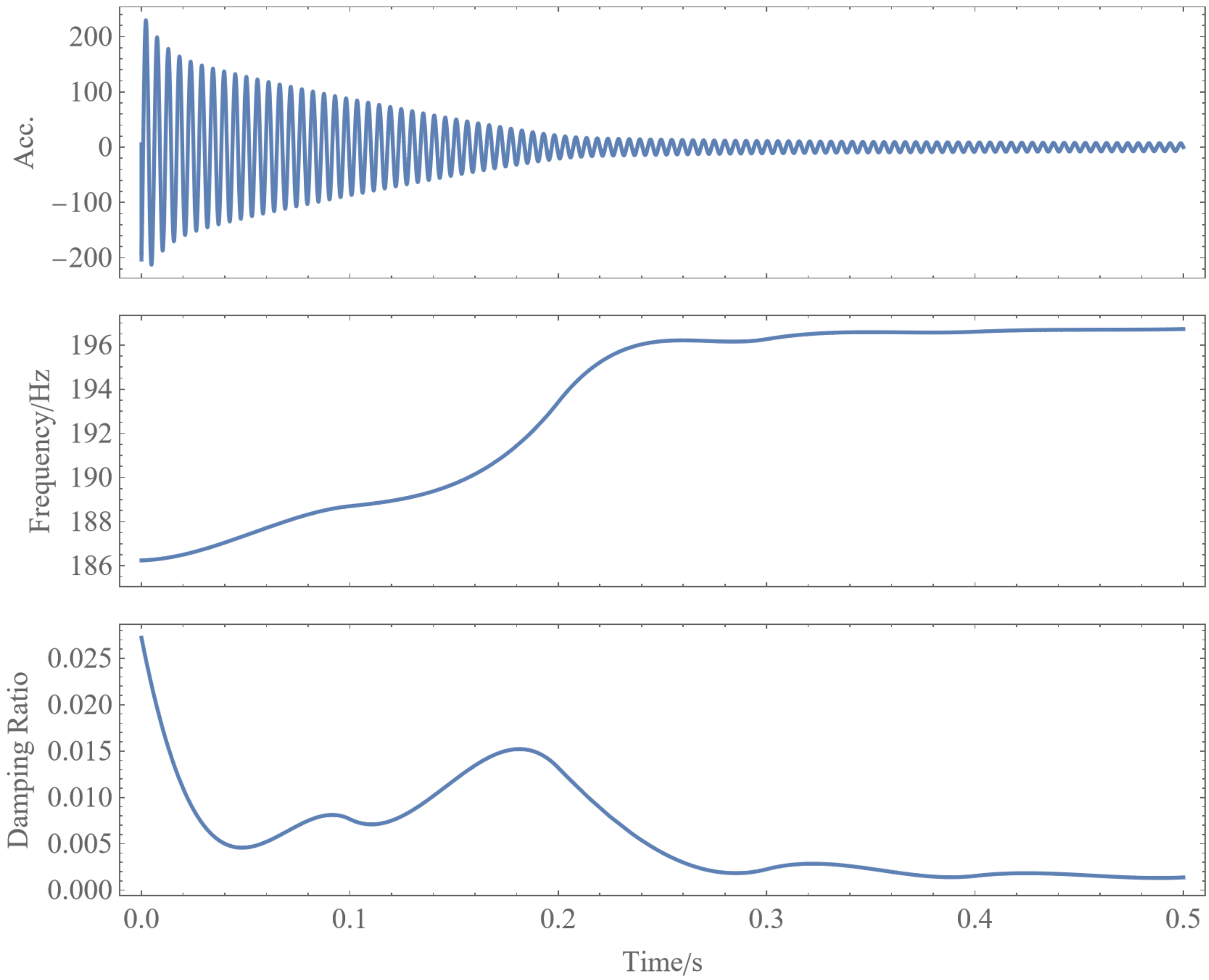

Figure 9. The instantaneous natural frequency and damping extracted for the first mode.

8 Discussion

An important assumption of this work has been that nonlinear modes exist and that they can be treated as independent. More understanding of such an assumption is needed. When is it not true? How does any coupling between nonlinear modes appear in practice? The fact that a Fourier transform exhibits peaks with very narrow bands suggests that a nonlinear system does have modes which can be treated with some form of superposition. This needs further consideration.

A Laurent series provides a complete mathematical model of a nonlinear structure in the frequency domain. It is completely general and extends the single degree of freedom model to any nonlinear system. Its generality for modelling nonlinear vibration warrants further investigation.

In a standard experimental modal analysis use is made of averaging and the results from several impulses are combined. This is possible for the nonlinear case but has not been included here. The difficulty is ensuring that averaging is done taking into account the amplitude of the vibration. Repeat measurements are always made however, the impulses applied may not be that similar and this must be taken into account.

The analysis presented above, and the use of a Laurent series, has been set in the Laplace s-plane. As the measured data has been sampled the more correct approach is to consider the z-plane. The Laurent series carries over to the z-plane and all the calculations performed have been done using the z-plane description. The formulation in the z-plane is straightforward.

Damping is one of the most difficult parameters to extract from an experimental modal analysis and is probably often nonlinear due to is source being due to friction. The method describe here is especially effective when damping is a key parameter being measured. Use of a shaker often interacts with a structure making the extraction of data difficult. Further, the signal processing procedures for extracting damping are not well developed for forced vibration. One method is the use a shaker to get large amplitudes of vibration and then to decouple the shaker and capture the subsequent vibration decay.

The use of reverse filtering is to be recommended as it clearly shoes the nature of the nonlinearity being investigated. It would be nice to go from the reverse filtered data to the individual modes. This may be possible and requires the process of deconvolution. This would slightly increase the decaying amplitudes and slightly change the instantaneous frequency and damping. An approximate method was reported by Goyder [4]. No exact method has yet been found.

In the above analysis the second mode turned out to be two close modes. Can such close modes be separated? The use of several measurement locations and excitation locations may enable this to be done. Further work developing the methods presented here are being investigated. The fact that two nonlinear close modes were identified is already a considerable achievement. The fact that they could then be separated from the other modes is also notable. The fact that the two close modes are not separated from each other is disappointing but not yet demonstrated as impossible. This is a difficult problem for linear systems so it is not unexpected that it is also difficult for nonlinear systems.

9 Conclusions

The following conclusions may be drawn

1. A nonlinear structure which exhibits a decay of vibration following an impulsive excitation may be investigated by breaking down the response into a sum of vibration decays. Each individual vibration decay may be associated with one nonlinear mode of the structure.
2. Each nonlinear mode can easily be analysed to determine instantaneous natural frequencies and damping. The key issue is to separate the modes in a vibration decay. A method for separating out the nonlinear modes has been presented.
3. It is recommended that reverse filtering is performed on a time history. The process of using a narrow band filter and reversing the data prior to filtering provides a very good method for revealing if a mode is nonlinear and the nature of the nonlinearity. It also provides a starting point for further analysis.
4. The Laurent series provides a complete mathematical basis for representing a nonlinear mode in the frequency domain. Its generality suggests it can be widely used.

Acknowledgements

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