A new scheme combining adaptive Kriging with adaptative variance-reduction using Gaussian mixture importance sampling

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Abstract
This article describes a new adaptive Kriging method designed to alleviate the limitations of other related approaches encounter in cases of extremely rare failure events. The main idea is to iteratively reduce both surrogate modelling error and sampling error. To do so the adaptive Kriging framework is associated with the multiple adaptive importance sampling scheme where the auxiliary distribution is iteratively built as a near-optimal Gaussian mixture. The estimator associated with he Gaussian mixture importance sampling is given as well as a stopping criterion based on both the estimated sampling and modelling error. The performances are finally illustrated on two benchmark problems.

1 Introduction

The probabilistic framework is the most common to approach uncertainty quantification (UQ) problems for industrial application. In this framework the uncertain parameters are modelled as random variables defined by a joint distribution aimed at being representative of their actual aleatory uncertainty. For the case of reliability analysis, the failure event is then defined as a function of a performance metric of the system studied and formulated in a so-called failure criterion. The framework also assumes a deterministic black-box relationship called a performance function is available linking the input parameters and the performance metric (e.g. a physics-based model of the system). The reliability problem lies in the fact that the output performance is then a random variable with an unknown distribution defined as the image of the input joint distribution through the performance function. The goal of a reliability analysis is then to quantify the probability of occurrence of the failure event related to the unknown output distribution. In addition, in most cases the performance function is not known analytically and its evaluation is costly (experimental or computation costs). The aim of reliability methods is then to estimate accurately the probability of occurrence of the failure event using the least amount of calls to the performance function.

The most common methods of solving this type of problems are based on random sampling approaches where the probability of occurrence of the event is estimated statistically by evaluating its occurrence rate in a dedicated population. While being quite straightforward, these approaches induce a statistical estimation error, sometimes called sampling-error and simply due to the limits of inferring a general statistical quantity from a finite population. The methods use a so-called estimator allowing to approximate the quantify of interest from the statistical properties of the population, and the variance of this estimator can be used as an indicator of the sampling error. In the context of reliability analysis the size of the population is of crucial importance to limit the computation burden. Therefore many sampling methods have been proposed to find

the best compromise between population size and variance of the estimator. Well know examples include Monte-Carlo [1], importance sampling [2], subset-simulations [3, 4], line-sampling [5, 6] or directional sampling [7].

Another approach to reduce the computation burden of reliability analysis called surrogate modelling has gained increasing popularity in the last few decades. The idea is to replace the physics-based performance function by an analytical emulator previously calibrated to mimic its behaviour. The advantage is that surrogates are usually orders of magnitude faster to evaluate than the physical model. The failure probability associated with the surrogate model can then be estimated with regular sampling methods using large populations at a limited computation cost. Many mathematical objects have been proposed to build surrogates, the most famous include response surface approaches [8], support vector machine [9], neural networks [10], polynomial chaos expansion [11] or Kriging (i.e. Gaussian processes) [12, 13]. The calibration of the surrogate relies on a set of samples called a design of experiment (DoE) for which the actual physical model has to be evaluated. This is why the calibration represents the main computation burden of the approach. In addition it induces another error called the modelling error due to the imperfection of the surrogate and which depends on the quality of its calibration. The goal of surrogate approaches is therefore to find the best compromise between the extent of the DoE and the modelling error associated with the failure probability.

Among the above-listed surrogate models, Kriging provides a useful feature in the shape of a localised estimation of its own accuracy. This estimation allowed the development of so-called adaptive-Kriging approaches where the goal is to attractively build a near-optimal DoE. The general idea stems from the realization that all areas of the input space do not have the same contribution to the estimation of the failure probability. Therefore the goal is to refine the calibration of the surrogate in areas of high contribution and avoid spending computation efforts calibrating areas of low contribution. This process is done iteratively using a so-called learning function classifying the candidate samples to select the most promising one to be added to the DoE. This approach was first introduced in the context of global optimization through the EGO method [14] and further popularized for the reliability analysis with the AK-MCS method [13]. For a more detailed overview of the adaptive-Kriging (AK) approaches the reader is referred to the work of Moustapha et al. [15]. This framework allows the algorithm to efficiently reduce the modelling error but when associated with a sampling method the sampling error then acts as an upper bound of the precision of the method. It is generally assumed that the sampling error will be negligible compared to the modelling error, but in cases of rare failure events this assumption can be wrong. Moreover, concluding regarding the validity of this assumption might only be possible after spending considerable computation efforts calibrating the surrogate on an ill-fitted population. These limitations motivated the development of a new adaptive Kriging method aimed at iteratively reducing both the modeling and the sampling error simultaneously and converging toward the failure probability no matter how rare the failure event.

The paper is organised as follows: section 2 presents the method in details, in section 3 the method is applied on two benchmark problems and the results are compared with a reference method from in the literature. The paper finally is concluded in section 4 with a brief summary and discussion.

2 Description of the method

The specificity of the proposed method lies on the way its population is build. As mentioned in the introduction the challenge with sampling-based reliability methods is to find a good compromise between the size of the population and the variance of the estimator. With respect to this compromise AK approaches can be sorted into two categories. The first one includes the vast majority of methods and consists in de-correlating the reduction of the sampling error with the reduction of the modelling error. The idea is either to run a preliminary analysis to define a semi-optimal population and then run the AK procedure to optimize the calibration of the surrogate, or to start by calibrating the surrogate and then generate an appropriate population. In both cases running the two optimization separately necessarily induce some loss of efficiency since the optimal population and the optimal DoE are deeply interdependent. These methods include AK-MCS [13], AK-IS [16], AK-MCMC [17], Meta-AK-IS^2 [18]. The second approach is to include the reduction of the sampling error in the AK procedure, coupled with the reduction of the modelling error. Much fewer methods

have been proposed within this framework, the most notable being probably the work of Balesdent et al. combining adaptive Kriging with non-parametric adaptive importance sampling [19].

The method proposed in this paper falls into the second category, combining adaptive Kriging with multiple importance sampling in a scheme where the optimal auxiliary distribution is approximated by iteratively building a Gaussian mixture. Its principle is that new Gaussian terms are iteratively added to the mixture based on a variance reduction criterion, new samples are then drawn from this Gaussian distribution and added to the overall mixture population. Both the refinement of the DoE and the auxiliary distribution are based on the Kriging surrogate estimations to keep the computation burden minimal. The method is finally associated with a stopping criterion considering both the expected modelling and sampling error.

In this section the method is presented in details starting with the multiple importance sampling estimator. The dedicated learning function allowing the definition of the DoE is discussed in section 2.2 and the sampling strategy allowing the definition of the auxiliary distribution is presented in section 2.3. Finally, in section 2.4 the stopping criterion is discussed with a flowchart of the method.

2.1 The estimator

In the proposed method the failure probability is ultimately estimated using a dedicated importance sampling scheme, where the auxiliary distribution is a potentially complex mixture of simple ones. This approach has been proposed before in various context [20, 21, 22] and will be referred to in this paper as multiple importance sampling.

Consider X a d-dimensional input random vector defined by the joint distribution f. The performance function of the system studied g is defined in such a way that a negative value of g indicates a failure. The failure probability P_f can be expressed analytically as follows:

$$P_f = \int_{\mathbb{R}^d} I(X) f(X) dX \tag{1}$$

with I a classification function associating a zero to safe samples and ones to failing samples. The principle of Monte-Carlo simulation is to approximate this integral numerically from a population $S_{MC} = \{X_1, ..., X_{N_{MC}}\}$ of $N_{MC} \in \mathbb{N}^*$ independent, identically distributed (i.i.d.) samples drawn from f. The associated estimator is the following:

$$P^{MC} = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} I(X_i) \tag{2}$$

Those well known developments (see e.g. [23]) are a good starting point to illustrate the importance sampling estimator. The idea behind importance sampling is to draw the population of i.i.d. samples from a different distribution than f and called an auxiliary distribution h. This distribution is chosen from a variance-reduction point of view and, in broad terms, is aimed at generating more failing samples than an Monte-Carlo population. Of course drawing the samples form a different distribution induces a bias in the estimator that has to be compensated for with a dedicated weight factor as follows:

$$P^{IS} = \frac{1}{N_{IS}} \sum_{i=1}^{N_{IS}} I(X_i) \frac{f(X_i)}{h(X_i)} \tag{3}$$

Before discussing the dedicated multiple importance sampling estimator, it is necessary to clarify the nature of the population used in this approach and the associated auxiliary distribution. The detailed sampling strategy is given in section 2.3 but the basic idea is as follows. The algorithm start with a very scarce Monte-Carlo population of n_1 samples. Then at each iteration k, a new Gaussian distribution h_k is picked from a variance-reduction criterion and n_k new samples are drawn from it. The overall aggregated population S_k is therefore composed of k sets of samples drawn from k different Gaussian distributions. It can be shown that

such a population is equivalent to an i.i.d. population drawn from the mixture distribution of all auxiliary ones weighted by the proportion of samples drawn from each (see e.g. [21, 24]). The corresponding mixture distribution can be defined as follows:

$$w_k = \frac{1}{N_k} \sum_{i=1}^{k} n_i h_i \tag{4}$$

with $N_k = \sum_{i=1}^{k} n_i$ the size of the overall aggregated population. It can finally be shown that the following expression represents an unbiased estimator of the failure probability with respect to the previously described population:

$$P^{MIS} = \frac{1}{N_k} \sum_{i=1}^{N_k} I(X_i) \frac{f(X_i)}{w_k(X_i)} \tag{5}$$

In this expression the classification function I is the one that requires the evaluation of the performance function, and is thus responsible for most of the computation burden of the analysis.

2.2 The adaptive Kriging scheme and the learning function

Kriging is a type of surrogate model that is based on the calibration of a Gaussian process. Even though a detailed description of Kriging surrogates would be out of scope for this work a brief overview is proposed in this section. In simple words, to every input vector X a Kriging surrogate associates a Gaussian random variable $\hat{g}(X) \sim N(\hat{\mu}(X), \hat{\sigma}(X))$ with $\hat{\mu}$ and $\hat{\sigma}$ respectively called the posterior mean and standard deviation to emphasize the Bayesian nature of Kriging calibration. For more details about Kriging models the reader is referred to useful dedicated material such as the manuals of Kriging toolboxes [25] or [26]. In the context on this work the posterior mean $\hat{\mu}$ will simply be considered as the Kriging estimation of the performance and $\hat{\sigma}$ is used as an indicator of the level of confidence associated with this estimation. The function that will be emulated is the performance function g, its surrogate counterpart being noted $\hat{g}$ and being equivalent to the posterior mean. From this function a surrogate counterpart can also be derived for the classification function called $\hat{I}$. The estimator defined in the previous section can then be estimated using the surrogate instead of the actual model as follows:

$$\hat{P}^{MIS} = \frac{1}{N_k} \sum_{i=1}^{N_k} \hat{I}(X_i) \frac{f(X_i)}{w_k(X_i)} \tag{6}$$

The motivation behind adaptive-Kriging approaches stems from the realization that all samples in the input sample do not contribute to the estimation of the failure probability in the same way. Therefore an efficient DoE would focus on refining the calibration of the surrogate on areas of great contribution and avoid losing computation time refining the calibration of areas of lesser influence. The adaptive-Kriging scheme therefore aims at taking advantage of the posterior standard deviation to iteratively select the samples of the DoE that have the highest expected contribution to the modeling error. The role of the so-called learning function is simply to sort the samples according to their contribution to the modelling error.

The learning function defined in [13] and associated with the AK-MCS method is used as a starting point for the definition of the ones dedicated to the present work. This learning function is based on the simple acknowledgement that the only modelling errors that contribute to the MC estimator are the "misclassifications" of samples, i.e. when a safe sample is tagged as failing by the surrogate or vise-versa. Since a failure is associated with a negative value of the performance function, this error is synonym of the surrogate being mistaken about the sign of the performance of a sample. Given the Gaussian nature of the the Kiging prediction, a convenient metric to assess the potential error associated with a prediction is the probability of misclassification U that can be efficiently formulated as follows:

$$U(X) = \Phi\left(\frac{-|\hat{\mu}(X)|}{\hat{\sigma}(X)}\right) \tag{7}$$

with Φ the standard normal cumulative distribution function. In order to define a dedicated learning function it is proposed to develop the expected modelling error as follows:

$$E\big(|\Delta P^{MIS}|\big) = E\big(|P^{MIS} - \hat{P}^{MIS}|\big) = E\left(\left|\frac{1}{N_k}\sum_{i=1}^{N_k}\frac{f(X_i)}{w_k(X_i)}\big(I(X_i) - \hat{I}(X_i)\big)\right|\right) \tag{8}$$

Equation (8) unfortunately provides an aggregated metric that does not inform us on the individual contribution of the samples with respect to the overall expected error. It is therefore proposed to focus on the following upper bound of the expected error which much easier to work with:

$$E\big(|\Delta P^{MIS}|\big) \leq \frac{1}{N_k}\sum_{i=1}^{N_k}\frac{f(X_i)}{w_k(X_i)}E\big(\big|I(X_i) - \hat{I}(X_i)\big|\big) \tag{9}$$

It can be shown (e.g. [27]) that $E\big(\big|I(X_i) - \hat{I}(X_i)\big|\big)$ is actually equivalent to the function U defined in equation (7). The upper bound can then be simplified as:

$$E\big(|\Delta P^{MIS}|\big) \leq \frac{1}{N_k}\sum_{i=1}^{N_k}\frac{f(X_i)}{w_k(X_i)}\Phi\left(\frac{-|\hat{\mu}(X_i)|}{\hat{\sigma}(X_i)}\right) \tag{10}$$

Finally, due mainly to computation limitations, Kriging surrogates have the tendency of underestimating their precision near well-calibrated areas which can result in the algorithm over-exploiting and not exploring enough the input space. In order to alleviate this issue a hyper-parameter p is introduced in the shape of a power factor affected to the posterior standard deviation. This parameters is only used for the learning functions of the algorithm and it is not considered in the evaluation of the failure probability or the stopping criteria detailed in section 2.4. The following learning function U^{MIS} is therefore defined as follows:

$$U_k^{MIS}(X_i) = \frac{f(X_i)}{w_k(X_i)}\Phi\left(\frac{-|\hat{\mu}(X_i)|}{\hat{\sigma}^p(X_i)}\right) \tag{11}$$

The above learning function is quite explicitly a generalization of the U function for populations of samples drawn from a mixture of auxiliary distributions. The term considering the probability of misclassification is present and weighted by the contribution of the sample to MIS estimator. Considering k iterations of the algorithm have been performed, the next sample to add to the DoE is called X_{k+1}^{DoE*} and is defined as follows:

$$X_{k+1}^{DoE*} = \arg\max_{X \in S_k} U_k^{MIS}(X) \tag{12}$$

2.3 The sampling strategy

As introduced in section 2.1 the sampling strategy deployed in this method can be seen as a type of importance sampling. In this context it is possible to derive an analytical expression of the optimal auxiliary distribution from the expression of the estimator given in equation (3). Unfortunately this expression depends on the performance function which is unknown analytically, the goal in this section is then to use the Kriging surrogate to iteratively build a near-optimal auxiliary distribution.

Other methods have been proposed in this direction for example using the non-parametric adaptive importance sampling method [28, 19]. A difficulty faced by those approaches is that they usually involve distributions that are hard to draw samples from and require the usage of advanced sampling methods such as

Markov-Chain Monte-Carlo sampling [29], which could cause implementation difficulties on their own. In this work it is proposed to introduce two simplifications to alleviate those difficulties. First, as introduced in section 2.1 the optimal distribution is approximated by a Gaussian mixture from which samples can be generated with classical semi-random number generators. Second, the problem of choosing the Gaussian terms of the mixture is further simplified by using the framework of the original mean-shift importance sampling. In other words, the problem is solved in the standard space and all individual Gaussian distribution is given a unit standard deviation. Therefore finding the next optimal Gaussian term of the mixture simply consists in finding the coordinates of the optimal mean.

In order to clarify the definition of this optimal mean coordinates, let us consider k iterations of the algorithm have been computed and develop the expression of the variance of the estimator defined in equation (5):

$$var\big(P^{MIS}\big) = \frac{1}{N_{k+1}} var\left(I(X)\frac{f(X)}{w_{k+1}(X)}\right) = \frac{1}{N_{k+1}}\left(E\left(I^2(X)\frac{f^2(X)}{w_{k+1}^2(X)}\right) - P_f^2\right) \tag{13}$$

The optimal distribution corresponds to a variance of zero and thus to the following equality holds:

$$E\left(\frac{I^2(X)}{P_f^2}\frac{f^2(X)}{w_{k+1}^2(X)}\right) = 1 \tag{14}$$

$w_{k+1} = \frac{If}{P_f}$ is a solution for of this equation and from the definition of the mixture given in equation (4) the following expression can be developed:

$$\frac{1}{N_{k+1}}\sum_{i=1}^{k+1} n_i h_i(X) = \frac{I(X)f(X)}{P_f} \tag{15}$$

It should be noticed that all h_i distribution are given at this stage of the algorithm, only the optimal h_{k+1}^* is to be determined. Its expression can then be isolated and developed as follows:

$$h_{k+1}^*(X) = \frac{N_k + n_{k+1}}{n_{k+1}}\frac{I(X)f(X)}{P_f} - \frac{N_k}{n_{k+1}} w_k(X) \tag{16}$$

Of course this expression depends on the classification function I and the actual failure probability P_f that are both unknown. The first intuitive simplification is to replace both by their surrogate counterpart respectively $\hat{I}$ and $\hat{P}^{MIS}$. The problem is then further simplified using the classical mean-shift importance sampling strategy which consist in choosing h_{k+1}^* as a Gaussian distribution of unit standard deviation and choosing its mean coordinates X_{k+1}^{Aux*} such that it maximizes equation (16). This strategy can be seen as choosing the Gaussian distribution such that its mode matches the mode of the optimal auxiliary. The coordinates of the optimal mean X_{k+1}^{Aux*} can then be defined by solving the following optimization problem:

$$X_{k+1}^{Aux*} = \arg\max_{X\in\mathbb{R}^d}\left(\frac{N_k + n_{k+1}}{n_{k+1}}\frac{\hat{I}(X)f(X)}{\hat{P}^{MIS}} - \frac{N_k}{n_{k+1}} w_k(X)\right) \tag{17}$$

The above expression allows us to define for each iteration a near-optimal Gaussian distribution considering the current calibration of the Kriging model and the current mixture distribution. n_{k+1} samples are drawn from this distribution and added to the overall population. The development of a complex strategy to define an optimal n_{k+1} is out of scope for this work. Instead it has simply been chosen to add a constant number of ten thousand samples per dimension and per iteration $n = d * 1e4$.

2.4 The stopping criterion

The algorithm is aimed at minimizing simultaneously both the modelling and sampling errors. It is therefore intuitively proposed to define a stopping criterion as the intersection of two criteria focused on each type of

error. The modelling error criterion is based on the convenient upper bound of the expected total error as defined in equation 11. A conservative threshold $E_{stop} = 4\%$ is then defined such that the modelling error criterion is satisfied when the following equation is satisfied.

$$\frac{1}{N_k}\sum_{i=1}^{N_k} U_k^{MIS}(X_i) \leq E_{stop} \tag{18}$$

The criterion related to the sampling error is based on the development of the variance of the estimator as defined in equation (13). This variance is then estimated using the surrogate counterparts of the classification function I and failure probability P_f. In addition the variance is estimated numerically from the available population S_k to avoid any additional computation burden.

$$\hat{var}(P_k^{MIS}) = \frac{1}{N_k}\left(\underset{X\in S_k}{\mathrm{E}} \left(I^2(X)\frac{f^2(X)}{w_k^2(X)} \right) - (\hat{P}^{MIS})^2 \right) \tag{19}$$

Finally the variance is normalized by the current estimate of the failure probability and compared to a threshold coefficient of variation $CoV_{stop} = 0.4\%$ such that the criterion is satisfied when:

$$\frac{\sqrt{\hat{var}(P_k^{MIS})}}{\hat{P}^{MIS}} \leq CoV_{stop} \tag{20}$$

The algorithm finally stops when both criteria are satisfied simultaneously. To conclude the presentation of the method a simplified flow chart is given in figure 1.

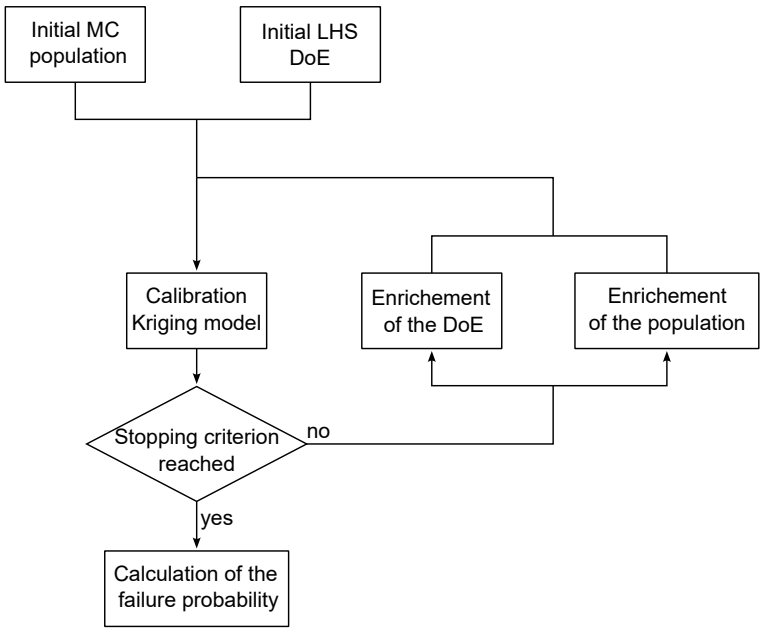


Figure 1: Simplified flowchart of the algorithm

3 Applications

In order to illustrate its performances, the proposed approach is applied to two benchmark analytical examples previously studied with the AK-MCS method [19] and well known for being troublesome to solve. In addition, it is proposed to observe how the performances scale with the failure probability. Therefore

each example function is considered with three sets of hyper-parameters inducing increasingly lower failure probabilities. For those application the Kriging surrogate is considered with a anisotropic Matern 5/2 covariance function. The initial design of experiment consist in a Latin hypercube design, centered on zero and containing $d + 3$ samples. The power parameter p defined in section 2.2 chosen equal to three.

Since the result of the analysis depends on random sampling its performances are stochastic. Each analysis is therefore performed ten times and the performance metrics are presented in terms of their mean value. The main metrics considered are the number of calls to the performance function, the covariance of the estimator and the error compared to a reference value of the failure probability. For comparison the results are compared with the AK-MCS method in terms of performances and a Monte-Carlo estimation of the failure probability is used as a reference. The implementation of AK-MCS considers an anisotropic Matern 5/2 covariance function, an initial design of experiment of ten Latin hypercube samples and a Monte-Carlo population of $min\left(\frac{500}{P^{MC}}, 5e6\right)$ samples. This population is chosen to result in a coefficient of variation of the estimator around 5%. The covariance of the proposed method is estimated as given in equation 20 and is estimated both for Monte-Carlo and AK-MCS as $C.O.V. = \sqrt{\frac{1-P^{MC}}{P^{MC} n_{MC}}}$.

It should be noted that the size of the population for AK-MCS implies a prior knowledge about the order of magnitude of the failure probability, which is not considered for the application of the proposed method. Unfortunately the performances of AK-MCS depend on the size of the population, therefore taking a default (very large) population would result in very high number of calls and very low coefficient of variation. For the sake of fairness in the comparison the size of the population is therefore chosen such that the sampling error of AK-MCS is reasonably low (around 5%). However for very small failure probabilities such an approach would induce prohibitory high computation times and is therefore capped at $5e6$ samples.

3.1 A function with four failure modes

The first example is a a well known function with four failure modes, designed to challenge the methods relying on identifying a design point. This examples is in dimension two with both independent random inputs $x_{1,2}$ following a standard normal distribution. The performance function reads as follows:

$$\forall x \in \mathbb{R}^2, g(x_1, x_2) = min \begin{cases} k_1 + \frac{1}{10}(x_1 - x_2)^2 - \frac{1}{\sqrt{2}}(x_1 + x_2) \\ k_1 + \frac{1}{10}(x_1 - x_2)^2 + \frac{1}{\sqrt{2}}(x_1 + x_2) \\ x_1 - x_2 + \frac{k_2}{\sqrt{2}} \\ -(x_1 - x_2) + \frac{k_2}{\sqrt{2}} \end{cases} \tag{21}$$

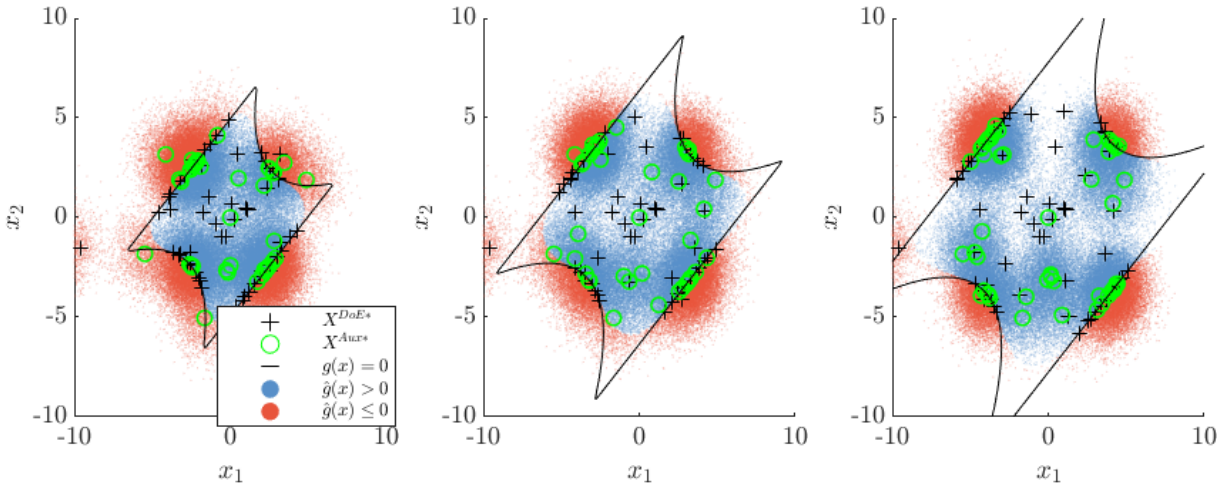
with k_1 and k_2 two parameters that can be chosen by the user. In this application three cases will be considered with respectively $(k_1, k_2) = \{(3.5, 7), (4.5, 9), (5.5, 11)\}$ and corresponding to increasingly smaller failure probabilities.

The performance of the proposed method are first illustrated on figure 2 with the converged mixture population and estimated limit-state for each case. The figure displays in blue (respectfully red) samples that are expected to be safe (respectfully failing) according to the Kriging surrogate. In addition the samples of the DoE are displayed with black crosses and the centers of the Gaussian auxiliaries with green circles. It can be seen that the algorithm successfully explores the input space and locates every major failure mode on all cases. As a frame of reference the three cases are respectfully associated with failure probabilities of around $e-3$, $e-5$ and $e-7$.

The performances are further illustrated in figure 3 with the convergence of the estimated failure probability compared with the one estimated with Monte-Carlo. The figure displays as a blue surface the amplitude of the estimated failure probabilities over the ten applications of the method. It can be seen that the method very efficiently and consistently converges toward the correct value, and that the order of magnitude of the failure probability has little influence on the pace of convergence.

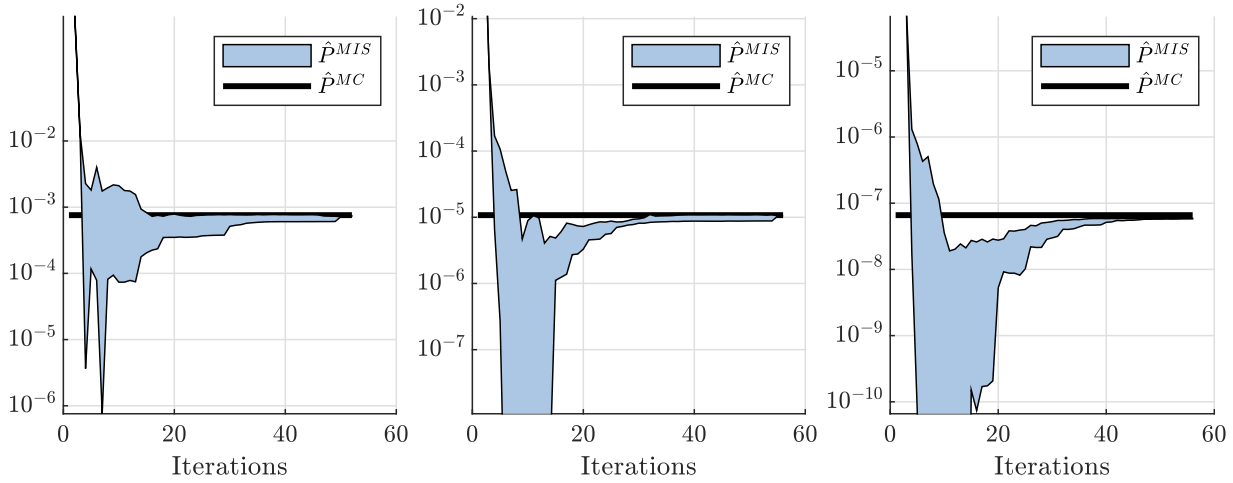
Finally the performances of the method are quantitatively compared with those of the classical AK-MCS method on table 1. It should be noted that for the third case the failure probability is too low to allow AK-

MCS for converging with the considered population size, the corresponding results are therefore left blank. The exhibited performances are very good compared with AK-MCS with two to three times fewer calls to the performance function, a very small coefficient of variation and very good accuracy (less than 3% error). The results are very consistent on the three cases considered with a very limited influence of the order of magnitude of failure probability. It is also noticeable that the modeling error and coefficient of variation are consistently lower than the threshold value chosen in section 2.4.



(a) $k_1 = 3.5,\ k_2 = 7$ (b) $k_1 = 4.5,\ k_2 = 9$ (c) $k_1 = 5.5,\ k_2 = 11$

Figure 2: Illustration of the converged population and Kriging calibration of the limit state for example 1 on the three cases a, b and c



(a) $k_1 = 3.5,\ k_2 = 7$ (b) $k_1 = 4.5,\ k_2 = 9$ (c) $k_1 = 5.5,\ k_2 = 11$

Figure 3: Convergence plot of the estimated failure probability compared with the Monte-Carlo reference for example 1 on the three cases a, b and c

3.2 Response of a non-linear oscillator

The second benchmark example is the response of a non-linear undamped oscillator with a single degree of freedom as illustrated on figure 4. The problem is in dimension six with all random parameters taken independent and defined by normal distributions as detailed in table 2. An extra parameter k_3 is introduced

Table 1: Results of the simulations on example 1

	method	$\hat{P}_f$	$\Delta P_f[\%]$	C.O.V.	N_{calls}
case1	MCS	$7.58e-4$	–	0.26	$2e8$
$k_1 = 3.5$	AK-MCS	$8.16e-4$	8.02	4.30	150
$k_2 = 7$	Proposed method	$7.40e-4$	2.99	0.25	49.2
case 2	MCS	$1.08e-5$	–	2.15	$2e8$
$k_1 = 4.5$	AK-MCS	$1.06e-5$	2.44	13.7	124
$k_2 = 9$	Proposed method	$1.06e-5$	2.67	0.29	54.4
case 3	MCS	$5.95e-8$	–	2.91	$2e10$
$k_1 = 5.5$	AK-MCS	–	–	–	–
$k_2 = 11$	Proposed method	$5.86e-8$	1.56	0.32	55.4

to artifically adjust the rarity of the failure event. Three cases are considered corresponding to respectively $k_3 = \{0, 0.5, 1\}$. The performance function reads as follows:

$$g(c_1, c_2, m, r, t_1, F_1) = k_3 + 3r - \left| \frac{2F_1}{m\omega_0^2} sin\left(\frac{\omega_0 t_1}{2}\right) \right| \tag{22}$$

with c_1 and c_2 the stiffness of the springs, m the mass, F the applied load, Z the position of the mass, r the maximum elongation before one of the springs yields and $\omega_0 = \sqrt{\frac{c_1+c_2}{m}}$.

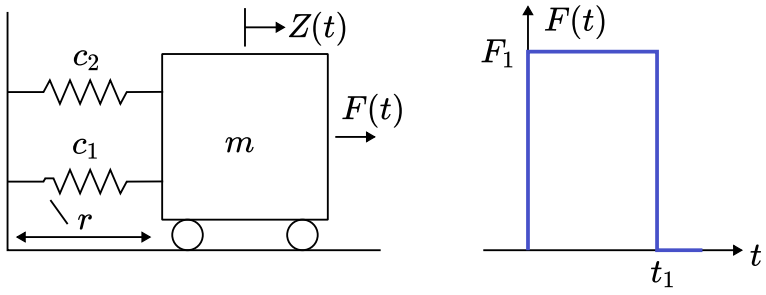


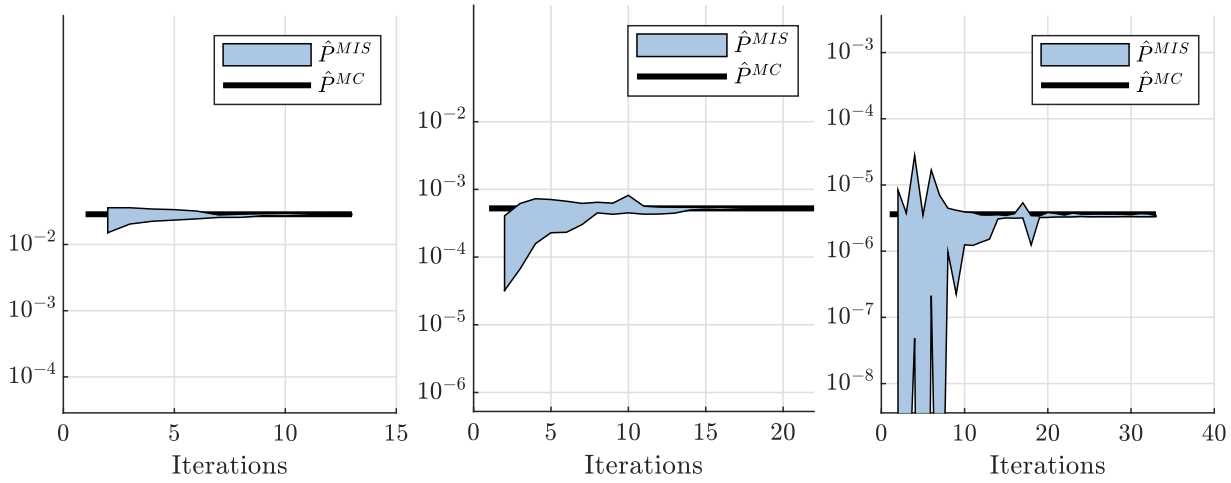
Figure 4: Illustration of the nonlinear SDOF oscillator and pulse load

Table 2: Parameters of the distributions for example 2

Variable	P.D.F	Mean	Standard deviation
m	normal	1	0.05
c_1	normal	1	0.1
c_2	normal	0.1	0.01
r	normal	0.5	0.05
F_1	normal	1	0.2
t_1	normal	1	0.2

The performances are illustrated on figure 5 with the convergence of the failure probability estimation. The amplitude of the estimation over the ten performed analysis is displayed with a blue surface. It can be observed that on all three cases the method efficiently converges toward the correct value with a limited influence of the order of magnitude of the failure probability.

The results are finally compared quantitatively with those of the reference AK-MCS method on table 3. The conclusions are similar than the ones drawn from example 1. On the first two cases AK-MCS is performing



(a) $k_3 = 0$ (b) $k_3 = 0.5$ (c) $k_3 = 1$

Figure 5: Convergence plot of the estimated failure probability compared with the Monte-Carlo reference for example 2 on the three cases a, b and c

as intended with a good accuracy (around 5% error) coefficient of variation (around 4%) and a reasonable number of calls to the performance function. On those two cases the proposed approach displays far superior performances than AK-MCS with a consistently better accuracy, coefficient of variation and about half the number of calls to the model.

On the last case the capped population of $5e6$ samples is not enough for AK-MCS to reach a sufficient accuracy. Therefore the algorithm terminates prematurely and, even though it only required sixteen calls to the performance function it reached an error and coefficient of variation of about thirty percent. On the same example the proposed approached exhibit good performances comparable to the other two cases.

Table 3: Results of the simulations on example 2

	method	$\hat{P}_f$	$\Delta P_f[\%]$	C.O.V.	N_{calls}
case1 $k_3 = 0$	MCS	$2.86e-2$	–	$4.12e-2$	$2e8$
	AK-MCS	$2.98e-2$	4.40	4.31	35
	Proposed method	$2.76e-2$	3.57	0.22	17.7
case 2 $k_3 = 0.5$	MCS	$5.24e-4$	–	0.31	$2e8$
	AK-MCS	$4.87e-4$	6.40	4.62	41
	Proposed method	$5.22e-4$	1.36	0.22	24.5
case 3 $k_3 = 1$	MCS	$3.27e-6$	–	3.91	$2e8$
	AK-MCS	$2.20e-6$	32.7	30.2	16
	Proposed method	$3.36e-6$	2.61	0.24	31.7

4 Conclusions

A new reliability method has been proposed in the framework of adaptive-Kriging approaches. This method is aimed at alleviating the limitations of most classical adaptive-Kriging approaches in cases of rare failure events. To do so the proposed approach seeks to attractively reduce both the modelling and sampling errors within the active learning scheme. The generation of the random is therefore done adaptively following a Gaussian mixture distribution, a dedicated learning function and estimator are given as well as a tailored and comprehensive stopping criterion.

The performances are illustrated on two classical benchmark problems and compared with the classical AK-MCS method. Each problem is studied through three cases with increasingly lower failure probabilities to

illustrate how the performances scale with the rarity of the failure event. On both examples and all cases the proposed approach converged much faster than the reference AK-MCS methods while providing comparable accuracy. In addition the proposed approach does not require any prior knowledge about the problem studied while the performances of AK-MCS as shown in the analysis correspond to a best-case scenario where the population size could be tailored to the problem. The analysis also highlighted how well the performances scale with the rarity of the failure.

Overall the methods shows very promising results with many advantages such as very few hyper-parameters, no prior knowledge needed and very good performances even on troublesome problems. However several limitations still remain such as the poor performances of Kriging models on high dimensions. In addition there exist opportunities for further improvement especially with the sampling part of the algorithm which takes many simplifying assumptions. A quite straightforward lead for future work would be to not only look for the optimum mean of the distribution but also include the standard deviation and the number of samples drawn for the auxiliary. Adding these two parameters to the optimization would allow for more flexibility in the definition of the Gaussian mixture which could result in a better approximation of the optimal auxiliary and a faster convergence.

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