Best practices in teaching flexible multibody dynamics

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Abstract

Using techniques from flexible multibody dynamics, it is possible to simulate the motion of mechanical systems in which bodies are subjected to large rigid body motions. Assuming that a body's local elastic deformation remains small, one can use linear finite element models to describe a body's flexible behavior. A dedicated course on flexible multibody dynamics was introduced at the University of Twente in 2016 successfully. In this article, the authors discuss the learning objectives of this course, its setup and examination. It is explained how theoretical lectures and practical working classes for numerical implementation can be organized as parallel activities by the means of 7 efficient weekly learning modules. For each module, important aspects with regards to study success or student motivation are explained. In this way, the authors wish to contribute to the successful teaching of flexible multibody dynamics by the engineering dynamics communities at other institutes.

1 Introduction

The field of flexible multibody dynamics (FMBD) is concerned with the simulation of mechanical systems that consist of multiple bodies undergoing large rigid body motions. The elastic deformation of individual bodies is integrated in the dynamic analysis and is solved simultaneously. This allows for more realistic simulation of the motion of the systems, and deformations and stresses that occur during these motions.

The floating frame of reference formulation, or simply the floating frame formulation (FFF) is a powerful formulation for the purpose of FMBD simulations. In this formulation, the rigid body motions of a body are described by the absolute motions of a coordinate frame that floats along with the body. Elastic deformation of the body are described locally, relative to the floating frame. If the local elastic displacements and strains remains small, linear elastic models can be used to describe this.

For simple body geometries analytical models, obtained from for example beam or plate theory can be used to describe the flexible behavior of bodies. For more complex body geometries linear finite element (FE) models can be used. This is possibly the most important benefit of the FFF: linear FE models of individual components can be reused in the FMBD simulation of an entire system. Moreover, the FFF allows to use powerful model order reduction techniques to be used on the linear FE models, to increase computational efficiency.

In recent years, the academic staff of the University of Twente (UT) that is involved with FMBD has observed an increasing interest in FMBD from industrial partners that see potential benefits in the more realistic simulation of their systems' dynamics. Also the topic is found very interesting by students in the Mechanical Engineering (ME) Master programme. For this reason, a dedicated course on FMBD was introduced at the UT in 2016.

At the UT, the ME programme already offers courses on applied numerical methods, such as the Finite Element Method for computational solid mechanics and the Finite Volume (FV) Method for computational fluid mechanics. Similar to these courses, the FMBD course has two main learning goals:

1. At the end of the course, students should understand the relevant theoretical background of the numerical methods used.
2. At the end of the course, students should be able to implement this theory numerically, in order to perform simulations.

Because the FE method and FV method have been part of the ME curriculum for long, experience has shown that it is possible to design such a course in roughly two ways: One option is to place learning goals 1 and 2 in series, first treating theoretical background and then applying it. Alternatively, one could place learning goals 1 and 2 in parallel, such that every piece of new theory is applied immediately.

From experience, it is known that the first option is well-suited for a course that is spread out over a longer period of time (a semester of 20 weeks) in which the weekly study load is relatively low. In the theoretical part of the course, there is sufficient time between lectures to process (or “digest”) the theory and in the application part of the course, there is sufficient time between working classes for debugging self-written codes or going through tutorials of commercially available software packages.

For a course that is taught in a shorter period of time (a quartile of 10 weeks), in which the weekly study load is relatively high, the second option is usually preferred. Because of time restrictions, students must be given the opportunity to start writing their own code or start making their own simulation model early in the course, in order to produce useful results by the end of the course. In terms of course organization, this means that lectures on the theory and working sessions on the application must be well-balanced.

At the time of writing this article, the ME Master programme at the UT is quartile-based, which is why the FMBD course was developed using the parallel setup discussed above. During the course development, the authors found that many classic textbooks on FMBD, that were considered to use for the course, start with the theoretical description of a general flexible multibody system, before treating the numerical implementation and other practical topics. Following the outline of such textbooks makes it hard to organize lectures on theory and working classes for numerical implementation as parallel activities.

For realizing time-efficient teaching of FMBD, the authors have developed a new framework using 7 weekly learning modules. Each module consist of relevant theory and a working class assignment for numerical implementation. After five years of continuous improvements, the authors now share their best practices in teaching FMBD. The main purpose of this article is to inspire academic staff of other universities in setting up courses in FMBD and to exchange ideas on how to teach this topic effectively.

The outline of this paper is as follows: In Chapter 2, the general overview of the course is discussed. It is explained which topics are covered in which order and which topics are intentionally left out. In Chapter 3, an overview of the 7 weekly learning modules is presented. The paper concludes with some final remarks and recommendations.

2 Course overview

The study load for the FMBD course is 2.5 European credits (EC), which corresponds to 70 hours. Each of the 7 learning modules requires roughly 8 hours of work: 2 hours for a lecture, 2 hours for a working class and 4 hours of self-study (studying theory and finishing the numerical implementation). The remaining 14 hours can be used for preparation of the final exam.

The main objective of the FMBD course is to give students a fundamental understanding of the floating frame formulation. For this, two important aspects of FMBD are identified:

1. One needs a fundamental understanding of the relevant kinematics and dynamics of a multibody system: which generalized coordinates are used, how to formulate kinematic constraint equations and how to determine the system’s nonlinear equations of motion.
2. One needs a fundamental understanding of how (reduced) linear FE models of individual bodies are integrated in the simulation of the entire system.

It was decided that conveying the above aspects as clearly as possible is the main purpose of the course. From aspect 1, it becomes clear that for FMBD problems the combined set of kinematic constraints and equations of motion form a set of differential-algebraic equations. The fact that this differs from the set of ordinary differential equations encountered in linear FE models is an important understanding. From aspect 2, it becomes clear what are the strengths and benefits of the FFF for the purpose of solving FMBD problems in comparison to nonlinear FE formulations.

In order to be able to put as much focus on the above aspects as possible, other related aspects or in-depth treatment of some topics is intentionally not treated in this course. The most important restrictions made are:

- All kinematic constraints are holonomic.
- All problems are two-dimensional. The additional complications of three-dimensional problems, such as the unique description of rotations in 3D is only briefly addressed. Students with a particular interest in applications in robotics are referred to a course that uses a formulation based on skew theory.
- Standard numerical time integration schemes are used. Students are referred to a course on the fundamentals of numerical methods for more information.
- It is assumed that a linear FE model for each flexible body is available in the form of a mass, damping and stiffness matrix. For how to create a proper FE model, students are referred to a course dedicated to the FE method.
- The concept of model order reduction is discussed briefly. Only the Craig-Bampton method is discussed. For more advanced techniques, students are referred to a course dedicated to model order reduction.
- No commercially available software packages are used.

Each weekly learning module consists of a lecture in which theory is discussed. This is followed by a working class in which students must implement the theory numerically in Matlab or Python (depending on the student's preference). For these purposes a standard slider-crank mechanism is used as a running example. To be able to focus on the relevant theory of a module, students are provided with a script that only requires to finish a designated section within the code. In this way, it is prevented that a lot of time is lost on the programming and debugging of parts of the code that are not directly related to the essence of FMBD.

Once the working class assignment is completed, students are asked to reuse their code for the simulation of a more complex multibody system of their own choice. Throughout the history of this course students came up with a wide variety of applications such as rehabilitation robots, excavators and cranes, suspension systems for bikes and cars, fairground attractions, drawbridges and water locks, deployment mechanisms for solar panel arrays and aircraft landing gears, wind turbines and many more.

To allow students to develop their numerical code incrementally, it is found best to start from simplified special cases and then expand towards the general framework: First, the kinematics of rigid multibody systems is discussed. This is then expanded to the dynamics. Flexibility is introduced using reduced order FE models of individual components. Finally, the constrained equations of motion of a set of general elastic bodies is considered.

It is the authors' experience that the most important benefit of this course outline is that the students remain very motivated throughout the course. Each combination of lecture and working class makes immediately clear what type of problem one is able to solve. This convinces students that performing FMBD simulations is *manageable*: it only takes a few hours to produce results that one was unable to produce before using the slide-crank system. To reuse this code on a problem of their own choice, ensures that the students perceive the course as *relevant*: they can apply this knowledge on an application that has their personal intrinsic interest.

The assessment of the course is by a final written exam. In order to be accepted for the final exam, the students must present the simulations that they performed on the FMBD system of their own choice. If this work is judged as sufficient on the most important aspects of the course, a student is admitted to the final exam, which determines the final grade. The written exam contains questions about relevant equations, modelling procedures and solution procedures that a student definitely will encounter during the development of their FMBD code. Because students only have permission to take the final exam if their final assignment is judged as sufficient, the passing rate of the written exam is relatively high.

3 Weekly learning modules

In this chapter, the theoretical contents of the 7 weekly learning modules is briefly discussed. For experts in the field of FMBD, this should be sufficient information to get a proper idea of which topics are treated in which order. For every module, important aspects with regards to study success and student motivation is provided. An overview of the learning modules is as follows:

3.1 Position analysis

For a rigid multibody system, the number of generalized coordinates that is required to uniquely define the configuration of the system relates to the number of bodies in the system. Due to kinematic constraints, the number of degrees of freedom of the system is less than the number of generalized coordinates:

$$N_{DOF} = N_q - N_c \tag{1}$$

in which N_{DOF} is the number of degrees of freedom, N_q the number of generalized coordinates and N_C the number of independent kinematic constraints. A vector function of N_C holonomic kinematic constraint equations $\boldsymbol{C}_K$ can be formulated in terms of the generalized coordinates $\mathbf{q}$ in the following form:

$$\boldsymbol{C}_K(\mathbf{q}) = \mathbf{0} \tag{2}$$

For the cases of a rotational joint and a translational joint, it explained how to formulate the corresponding kinematic constraint equations. A vector of N_{DOF} driving constraints $\boldsymbol{C}_D$ is added by prescribing the desired motion as a function of time explicitly:

$$\boldsymbol{C}_D(\mathbf{q}, t) = \mathbf{0} \tag{3}$$

The total set of constraint equations, is the assembly of the kinematic constraint equations $\boldsymbol{C}_K$ and the driving constraint equations $\boldsymbol{C}_D$. These are N_q nonlinear algebraic equations for N_q unknown generalized coordinates:

$$\boldsymbol{C}(\mathbf{q}, t) = \begin{bmatrix} \boldsymbol{C}_K(\mathbf{q}) \\ \boldsymbol{C}_D(\mathbf{q}, t) \end{bmatrix} = \mathbf{0} \tag{4}$$

Because of the nonlinear nature of the constraint equations (4), an analytical solution is difficult to obtain. Instead, a solution is obtained numerically using the iterative Newton-Raphson method. At the end of this module, students are able to define the total set of constraint equations for commonly encountered mechanisms such as the slider-crank mechanism, the four bar mechanism, the quick-return mechanism and so on. After numerically solving the set of constraint equations, the configuration of the system as a function of time can already be animated. Producing such animations is attractive, because it shows students that quite nice results can be obtained fast for realistic systems, which motivates them to continue.

It is interesting to explain that for systems with simple kinematics (open kinematic chain / tree structure) the constraint equations (4) can be solved analytically quite easily. Students probably did so in earlier courses on engineering dynamics or mechanical vibrations. For instance, by expressing all generalized coordinates of a slider-crank system in terms of the driving angle, one demonstrates that for systems with more complex kinematics (closed kinematic chains), an analytical solution may still exist, but that because of the square roots or inverse sines or cosines, this analytical solution is not very attractive. This helps students to understand why a numerical solution method is relevant to study.

3.2 Velocity analysis and acceleration analysis

The set of constraint equations (4) is differentiated once with respect to time to obtain the velocity equation:

$$\boldsymbol{C}_{\mathbf{q}}\dot{\mathbf{q}} = \boldsymbol{\nu}(\mathbf{q}, t) \tag{5}$$

in which $\boldsymbol{C}_{\mathbf{q}}$ is the Jacobian of the constraint equations. Differentiating once more with respect to time yields the acceleration equation:

$$\boldsymbol{C}_{\mathbf{q}}\ddot{\mathbf{q}} = \boldsymbol{\gamma}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{6}$$

It is explained that because equations (5) and (6) are linear in terms of the velocities and acceleration respectively, the velocities and accelerations can be determined in each configuration exactly, without introducing any numerical error. Hence, when the configuration of a system is known, or solved with the Newton-Raphson method with sufficient accuracy, calculating the velocities and accelerations is straightforward. Based on this, students are able to conclude the kinematic analysis of rigid body systems adding plots and visualizations of velocities and accelerations to the position analysis.

At this point it is good to look back to the students earlier course in engineering dynamics in which they were able to perform a velocity and acceleration analysis on rigid multibody systems of which the configuration is given. Students can understand now, that the difficulties are really on the position level and not on velocity or acceleration level. It is very useful to go back to the book of an engineering dynamics course and summarize that all problems contain terms as “at this point in time”, “in the configuration shown”, “at the point of slip”, “at the moment the cable breaks”.

3.3 Constraint forces

The equations of motion in Lagrange-multiplier form are derived:

$$\mathbf{M}\ddot{\mathbf{q}} + \boldsymbol{C}_{\mathbf{q}}^{T}\boldsymbol{\lambda} = \mathbf{Q}_{a} \tag{7}$$

in which $\mathbf{M}$ is the mass matrix of multibody system, $\boldsymbol{\lambda}$ the vector of Lagrange multipliers and $\mathbf{Q}_a$ the vector of externally applied forces. It is explained that the Lagrange multipliers represent the generalized constraint forces. Because the system is still considered to be kinematically driven, the $\boldsymbol{C}_{\mathbf{q}}$ is square. Hence equation (7) can be solved for $\boldsymbol{\lambda}$:

$$\boldsymbol{\lambda} = \left(\boldsymbol{C}_{\mathbf{q}}^{T}\right)^{-1}(\mathbf{Q}_{a} - \mathbf{M}\ddot{\mathbf{q}}) \tag{8}$$

It is explained how the reaction forces and moments in each interface point of the system can be computed from the Lagrange multipliers. Also the required driving force or moment to realize the prescribed motion can be computed. At this point, it is good to emphasize that we have reached the end of kinematically driven systems. Based on a prescribed motion, we are able to solve for position, velocity, acceleration and constraint forces as a function of time.

During the lecture one can derive equation (7) theoretically, using for instance the principle of virtual work. It is attractive to combine this with deriving the equations of motion of a simple system, e.g. a compound pendulum, by drawing a free body diagram and writing down the Newton-Euler equations. By writing the terms related to the constraint forces in matrix-vector form, one can see the Jacobian matrix $\boldsymbol{C}_{\mathbf{q}}$ arising naturally. In this way, students can be taught that the Lagrange multipliers may represent constraint forces directly, or, depending on how one introduced the constraint forces in the free body diagram, that the Lagrange multipliers are linear combinations of the constraint forces. This makes the concept of Lagrange multipliers easier to absorb.

3.4 Kinetically driven systems

For kinetically driven systems, sometimes also referred to as dynamic systems, there are no driving constraints at all or the number of driving constraints is less than the number of degrees of freedom. This means that there are more generalized coordinates $\mathbf{q}$ than constraint equations $\boldsymbol{C}$. Consequently, the constraint equations and the equations of motion must be solved simultaneously. The constrained equations of motion in augmented form are derived:

$$\begin{bmatrix} \mathbf{M} & \boldsymbol{C}_{\mathbf{q}}^T \\ \boldsymbol{C}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_a \\ \boldsymbol{\gamma} \end{bmatrix} \tag{9}$$

At each time step, the system matrix and right hand side vector of equation (9) is determined. Then, the equation is solved for the generalized accelerations $\ddot{\mathbf{q}}$ and Lagrange multipliers $\boldsymbol{\lambda}$. The generalized coordinates $\mathbf{q}$ and generalized velocities $\dot{\mathbf{q}}$ at the next time step are determined using a numerical time integration scheme.

It is explained that because equation (9) only satisfies the constraint equations at acceleration level, it must be checked that the generalized coordinates at the next time step satisfy the constraint equations $\boldsymbol{C}$ with sufficient accuracy. If this is not the case, the generalized coordinates must be corrected. To this end, the generalized coordinates are partitioned in the independent generalized coordinates $\mathbf{q}_i$ and dependent generalized coordinates $\mathbf{q}_d$. The independent generalized coordinates that result from the numerical time integration are accepted. The dependent generalized coordinates are updated iteratively, using the Newton-Raphson method, until the constraint equations are satisfied with sufficient accuracy.

At the end of this module, students should understand why the constrained equations of motion in augmented form must be solved using numerical time integration. Although the Newton-Raphson method is able to satisfy constraints with arbitrary accuracy, the solution will not be exact, due to the error in the numerical time integration. At this point, the dynamics of rigid multibody system is finished.

In previous modules, the students have computed the required driving moment that should be applied on a slider-crank system in order to have it rotate at a constant angular velocity. In this module, the students have successfully simulated the motion of a slider-crank system that is subjected to a prescribed driving moment. It is useful to apply the computed driving moment from module 3 as prescribed driving moment to see if the resulting motion is indeed a rotation with constant angular velocity. This also creates the possibility to attribute any differences to the numerical error caused by the time integration.

For students with a particular interest in control engineering, it can be useful to rewrite equation (9) to embedded form. In this, the generalized coordinates are partitioned in the dependent and independent generalized coordinates. By manipulating equation (9), it is possible to find a reduced set of equations in terms of an embedded mass matrix that is multiplied by the independent generalized acceleration only.

3.5 Model order reduction of linear FE models

It is assumed that a linear FE model of each flexible body in the FMBD system is available in the form a mass matrix, stiffness matrix and potentially damping matrix. Such FE models are typically already available indeed, as one would use such modes for a strength analysis that is based on estimated load cases. Consequently, these FE models can have a large number of degrees of freedom. In order to keep the computational costs of the FMBD simulations acceptable, model order reduction techniques can be used on each of the FE models. Because the kinematic constraints of the FMBD model are enforced at the interface points between bodies, one wants the reduced model to be accurate at these interface points. For that reason, the Craig-Bampton method is often used. In this method the boundary modes, also referred to as interface modes or static modes that follow from a Guyan reduction or static reduction are augmented with internal vibration modes that are determined by keeping all interface nodes fixed. Let the stiffness matrix $\mathbf{K}$ of an FE model be partitioned according to the boundary nodes (subscript b) and internal nodes (subscript i):

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \tag{10}$$

The boundary modes $\mathbf{\Phi}_b$ are computed as:

$$\mathbf{\Phi}_b = \begin{bmatrix} \mathbf{1} \\ -\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib} \end{bmatrix} \tag{11}$$

The internal vibration modes $\mathbf{\Phi}_i$ are solutions of the following eigenvalue problem:

$$[\mathbf{K}_{ii} - \omega^2\mathbf{M}_{ii}]\mathbf{\Phi}_i = \mathbf{0} \tag{12}$$

in which $\mathbf{M}_{ii}$ is the partition of the FE mass matrix related to the internal nodes. In the Craig-Bampton method, the reduction basis $\mathbf{\Phi}_{CB}$ that is used for the model order reduction is the combination of $\mathbf{\Phi}_b$ and $\mathbf{\Phi}_i$:

$$\mathbf{\Phi}_{CB} = [\mathbf{\Phi}_b \quad \mathbf{\Phi}_i] \tag{13}$$

When the floating frame is attached to one of the nodes of the FE model, it is possible to formulate 6 rigid body modes $\mathbf{\Phi}_{rig}$ easily. Now, the final reduction basis $\mathbf{\Phi}$ is defined as the set of rigid body modes and Craig-Bampton modes:

$$\mathbf{\Phi} = [\mathbf{\Phi}_{rig} \quad \mathbf{\Phi}_{CB}] \tag{14}$$

Reduced mass $\mathbf{M}_{red}$ and stiffness $\mathbf{K}_{red}$ matrices are computed as follows:

$$\mathbf{M}_{red} = \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}, \qquad \mathbf{K}_{red} = \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} \tag{15}$$

Based on these reduced matrices, coupling between the generalized coordinates that describe rigid body motion and the generalized coordinates that describe local elastic deformations is explained. For example, it is observed that the (3×3) top left partition of the reduced mass matrix $\mathbf{M}_{red}$ equals the rigid body mass matrix $\mathrm{diag}(m, m, I)$ and that many terms of the reduced stiffness matrix are zero, because $\mathbf{\Phi}_{rig}^T\mathbf{K} = \mathbf{0}$ and $\mathbf{K}\mathbf{\Phi}_{rig} = \mathbf{0}$. In fact $\mathbf{\Phi}_{CB}^T\mathbf{K}\mathbf{\Phi}_{CB}$ is the only nonzero partition of the reduced stiffness matrix.

For the example of the slider-crank mechanism, students first create a FE model based on a large number of beam elements. They determine the rigid body modes and Craig-Bampton modes and subsequently the reduced FE matrices. Visualization of the rigid body modes and Craig-Bampton modes helps to understand the nature of the reduction basis.

3.6 Flexible multibody dynamics

At this point, it is understood how to create a rigid multibody model and how to create a reduced linear FE model for each of the flexible bodies in a FMBD system. Now it is time to combine this knowledge and create the full FMBD model. This requires several steps.

First the constraint equations $\boldsymbol{C}$, the resulting Jacobian matrix $\boldsymbol{C}_{\mathbf{q}}$ and $\boldsymbol{\gamma}$ are modified to take the flexible behavior of bodies into account. For this purpose, the kinematics of an arbitrary point on a flexible body is presented completely. When taking flexibility of bodies into account, the set of generalized coordinates $\mathbf{q}$ is augmented: not only does it contain the absolute coordinates of a body's floating frame, but it now also contains the generalized coordinates corresponding to a body's Craig-Bampton modes, which are local coordinates, relative to a body's floating frame. For the cases of a rotational joint and a translational joint, it is explained how the constraint equations must be modified to take the flexibility into account.

Next, the constrained equations of motion in augmented form (9) can be updated to include the flexible behavior of bodies:

$$\begin{bmatrix} \mathbf{M} & \boldsymbol{C}_{\mathbf{q}}^T \\ \boldsymbol{C}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_a - \mathbf{Q}_e \\ \boldsymbol{\gamma} \end{bmatrix} \tag{16}$$

In this, $\mathbf{M}$ now represent the mass matrix of the entire system, which is the assembly of all bodies' reduced mass matrix $\mathbf{M}_{red}$ as determined from equation (15). $\mathbf{Q}_e$ is the vector of elastic forces, which is simply assembly of all bodies' reduced stiffness matrix $\mathbf{K}_{red}$ as determined from equation (15), multiplied with the generalized coordinates $\mathbf{q}$.

It is important to emphasize that the number and nature of kinematic constraint equations does not change when bodies are modelled as flexible. However, (many) more generalized coordinates are now involved in these kinematic constraints. From this, it is directly clear that for flexible systems, it is not possible to solve the position analysis using the Newton-Raphson method only.

The solution procedure of equation (16) is the same as for equation (9). In this, it is important to understand that the generalized coordinates related to a bodies' Craig-Bampton modes must be treated as independent generalized coordinates as well. This means that their value at a next time step, as computed during numerical time integration is directly accepted. Using Newton-Raphson iterations, the same dependent coordinates as before are updated to satisfy the constraint equations.

Equation (16) is not derived, but simply presented. The difference with the constrained equations of motion of a rigid multibody system is only small, such that it is easy to convince students that this form might be correct for flexible multibody systems. Omitting the derivation is a deliberate choice, such that students can already start implementing the full flexible multibody system without further delay. The numerical code is developed such that animations and plots of the flexible deformations during the large overall motion are easy to visualize.

3.7 Formal derivation of the equations of motion of a flexible body

The method presented in weekly learning module 6 is the fastest way to get a FMBD code up and running and the simulation results are satisfying for students. However, in constructing the constrained equations of motion in augmented form (16), it was just assumed that we can simply replace the mass matrix of the rigid system with the mass matrix of the flexible system and add the elastic forces on the right hand side. It was never formally proven that this is the correct equation of motion for a flexible system.

For that reason, in this last weekly learning module, the formal derivation of the equations of motion of a flexible body are presented. To this end, the kinematic expressions for an arbitrary point on a flexible body are reused from module 6. The principle of virtual work is used to establish relations for the virtual work due to inertia forces, virtual work due to internal elastic forces and virtual work due to externally applied forces. Upon careful differentiation of the kinematic relations, it can be shown that the constrained equations of motion (16) are not completely correct, because quadratic velocity forces $\mathbf{Q}_v$ are missing. The correct equation reads:

$$\begin{bmatrix} \mathbf{M} & \boldsymbol{C}_{\mathbf{q}}^T \\ \boldsymbol{C}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_a - \mathbf{Q}_e - \mathbf{Q}_v \\ \boldsymbol{\gamma} \end{bmatrix} \tag{17}$$

The quadratic velocity forces originate from the virtual work due to inertia forces. They include inertia coupling between the generalized velocities of the floating frame and the first time derivatives of the generalized coordinates corresponding to the Craig-Bampton modes. Because the elastic deformations remain small, the quadratic velocity forces can often be ignored, certainly in 2D. In 3D one needs to be careful in applications that involve high angular velocities, such as rotor dynamics problems.

The lecture in this weekly learning module contains the formal presentation of the kinematics and kinetics of the FFF, as one can find it in standard textbooks on this method. This also includes the formal presentation of inertia shape integrals. It is explained that it is more convenient to use the FE discretization directly to approximate these inertial shape integrals, than to try to compute them analytically and discretize later. In this way, students are able to compute an approximation of $\mathbf{Q}_v$ based on the FE mass matrix. This is included in the constrained equations of motion in augmented form (17), which is solved numerically. By comparing this result with the result obtained from solving (16), students can judge the relative importance of the quadratic velocity forces. For the standard slider-crank problem, it can be concluded that the quadratic velocity effects can be ignored. For most applications that are defined by the students themselves this is also the case.

4 Conclusion and recommendations

The course overview as presented in this article contain, in the author’s experience, the optimal way of teaching FMBD to students. The weekly modules are designed such that their study load is approximately equal and that each module consists of a clear theorical concept that can be discussed in a plenary lecture and a clear assignment to numerically implementation. Attending the lectures and working classes and finishing the assignment within the allocated time for self-study is possible, provided that students have some basic experience with programming in Matlab or Python.

Based on the list of restrictions made in Chapter 2, it should be understood that this course serves only as an introduction to FMBD. However, because of all these restrictions, it is possible to focus on the most important ideas and approaches. Consequently, if students are interested in FMBD and want to study it in more detail during an internship, research assignment or graduation project, it is relatively easy to refer them to relevant literature in which it is explained how to deal with more complex FMBD problems. Having completed this course, students will be able to make useful contributions to both internal projects at the university and external projects in industry.

The success of knowledge transfer during the weekly modules highly depends on students being on track. For that reason, it is highly recommended to emphasize at the start of the course that thorough understanding comes only after one has struggled with the numerical implementation themselves. Students are recommended to try the implementation of the slider-crank system individually and then work on a final assignment in small groups (2 or 3 students).

At the time of writing this article, the authors are in the process of creating relevant video lectures for each weekly learning module. Once this is done, they will be made publicly available online. At the same time, relevant Matlab and Python scripts will be shared with the engineering dynamics community. With this material, one could setup a similar course at other academic institutes with relatively low effort or allow students to study FMBD by self-study. In case of interest in this online material, please contact the corresponding author.